AN

ALGEBRAIC SYSTEM

OF

CONIC SECTIONS,

AND

OTHER CURVES.

BY J. M. F. WRIGHT, A. B.,

LATE SCHOLAR OF TRINITY COLLEGE, CAMBRIDGE; AUTHOR OF SOLUTIONS OF THE CAMBRIDGE PROBLEMS; SELF-EXAMINATIONS IN ALGEBRA; THE PRIVATE TUTOR, ETC.

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THE RIGHT HONOURABLE

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THE FOLLOWING WORK

IS,

WITH HIS LORDSHIP'S PERMISSION,

MOST RESPECTFULLY DEDICATED

BY

HIS LORDSHIP'S OBLIGED AND GRATEFUL SERVANT,

J. M. F. WRIGHT.

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Either co-ordinate of a curve, whose equation is of (n) dimensions, has n, or $n - 2$, or $n - 4$, &c., different values for

ERRATA.

Page	Line	
3	1	for *then*, read *the infinite areas*.
13	last but one,	for (2, 3), read (2, − 3).
17	7	for (a, b), read (a, b), (a', b).
23	2	for $=$ R, read $=$ R$'$.
25	15	for $\left(\mathrm{R}, \frac{\pi}{2}\right)$, read $\left(\mathrm{R}', \frac{\pi}{2}\right)$.
30	5	for $b =$, read $b = -$.
—	6	for (s, r), read (s, x), and for $\frac{b'' - b'}{a'' - a'}$, read $-\frac{b'' - b'}{a'' - a'}$.
—	7	for $\frac{a'' - a'}{b'' - b'}$, read $-\frac{a'' - a'}{b'' - b'}$.
31	2	for $\frac{b - b'}{\frac{a}{b} - \frac{a}{b'}}$, read $\frac{b - b'}{\frac{b}{a} - \frac{b'}{a'}}$.
—	7	for $\frac{a' b'' - a'' b'}{a'' - a'}$, read $-\frac{a' b'' - a'' b'}{a'' - a'}$.
—	8	for $\frac{a_{\prime} b_{\prime\prime} - a_{\prime\prime} b_{\prime}}{a_{\prime\prime} - a'}$, read $-\frac{a_{\prime} b_{\prime\prime} - a_{\prime} b_{\prime\prime}}{a_{\prime\prime} - a_{\prime}}$.
32	14	for $7x$, read x.
—	20	for $\frac{20}{13}$, read 20.
—	21	for $\frac{77}{13}$ read 9.
—	23	for $\left(\frac{20}{13}, \frac{77}{13}\right)$, read (20, 9).
33	last,	for cos. α, read cos. (s, s').
35	1	for 7 read 1; and for $\tan.^{-1} \frac{\frac{7}{3} - 1}{1 + \frac{7}{3}} = \tan.^{-1} \frac{4}{10} = \tan.^{-1} \frac{2}{5}$, read $\tan.^{-1} (-\frac{1}{2})$.
—	6	for $\frac{\frac{a}{b} - \frac{a'}{b'}}{1 + \frac{a}{b} \cdot \frac{a'}{b'}}$, read $\frac{\frac{b}{a} - \frac{b'}{a'}}{1 + \frac{b}{a} \cdot \frac{b'}{a'}}$.
40	14 and last,	for *common*, read *co* – ; and in the last line, for *common*, read *co* –
41	1	for *co-ordinates*, read *common ordinates*.

Page	Line	
—	2	for *point*, read *right line*.
—	4	for *is that*, read *is the co-ordinate*.
—	5	for *common* read *co*.
—	6	for *co*, read *common*.
—	11	for $a\,-$, read $a\,+$, and for $b\,-$, read $b\,+$.
42	11	for $a\,b' + a'\,b'$, read $a'\,b + a\,b'$.
46	16	for $r^2 +$ read $r^2 -$.
—	23	for x read x^2.
51	last but three,	for $r^2 +$ read $r^2 -$.
52	2	for $\frac{A}{2}$ read $-\frac{A}{2}$.
—	11	for $\frac{3}{2}$ read $-\frac{3}{2}$.
—	15	for cos. 50° sin. θ, read cos. 40° cos. θ; and for sin. 50 cos. θ, read sin. 40° sin. θ.
53	7	for $\frac{3}{2}$ read $-\frac{3}{2}$.
—	8	for $R' - \frac{3}{2}$ read $R' = -\frac{3}{2}$.
—	9	for $\frac{3}{2}$, read $-\frac{3}{2}$.
55	3 and 4,	for $r = R'$, read $R' \cos \alpha$, and also in line 7.
56	last but 1,	for *then*, read *these*.
57	5 from bottom,	for $2 \mp 4 \sqrt{2}\, 3$, read $3 \mp 6 \sqrt{2}\, 3$.
—	4 from bottom,	for $4\,1 \mp 4 \sqrt{2}\, 3$, read $4\,2 \mp 6 \sqrt{2}\, 3$, and make the like corrections in the following lines.
60	5	for $\frac{a' - a}{a\,b' - a\,b'}$ read $-\frac{a' - a}{a\,b' - a'\,b}$.
61	6	for $\left.\begin{matrix}\frac{a'}{a''} + \frac{b'}{b''} \\ \frac{a}{a''} + \frac{b}{b''}\end{matrix}\right\}$ read $\left.\begin{matrix}\frac{a'}{a''} + \frac{b'}{b''} = 1 \\ \frac{a}{a''} + \frac{b}{b''} = 1\end{matrix}\right\}$.
62	6 from bottom,	for $b' = \frac{R^2}{a}$ read $b' = \frac{R^2}{b}$.
63	2	for $-\frac{a}{b}$ read $\frac{a}{b}$.
126	4 from bottom,	for $a'^2\,b'^4$ in denominator, read $a'^4\,b'^4$.

PREFACE.

Analytical Geometry, considered in its important practical applications to every branch of Natural Philosophy, is, perhaps, the most interesting of all the subjects in the pure mathematics. Yet, until very recently, it has received but little cultivation in Britain. Whilst the great writers of the Continent, and even of America, have sedulously brought it to a great degree of perfection, applying its powers to the full development of the secrets of material nature, the authors of this country have almost wholly disregarded it. Whilst British mathematicians have still been occupied in unravelling the complex geometrical synthesis of Newton, from a pernicious, though natural, adoration of the great founder of all true philosophy, pertinaciously refusing to let in the indefinitely more powerful lights of Analysis, their rivals in France, Italy, Germany, and the United States have been enabled, by letting in the full flood of those lights, not only to see as far as Newton himself, and even to complete the researches into the celestial motions which, with all his acknowledged superhuman genius—" *Qui genus humanum ingenio superavit,*"—during one state of existence he could but imperfectly accomplish, but even to attempt, after quitting the theory of the motions of remote bodies, that more intricate one of the chemical, electrical, and other actions of the matter of which they

themselves, and all other terrestrial masses, are compounded.

A great change, however, has at length been effected. One of the greatest philosophers of this country, Professor Wallace of Edinburgh, thus expresses himself on the subject in a letter to the author of this work, which letter he has already been kindly permitted to insert in his periodical called 'The Private Tutor.' "It is gratifying," says the Professor, "to cultivators of the mathematical sciences, to see the great change that has taken place in the style of their science at the fountain-head—Cambridge, and the ardour with which it is now pursued. It was Woodhouse that commenced the reformation, for which honoured be his memory, &c."

Woodhouse, indeed, it was that led the way and kept a firm footing, despite the great opposition he met with from many who from their talents ought not to have yielded to the influence of such prejudices. So strong and violent was this opposition, venting itself in various pamphlets, that it produced as strong a re-action, giving occasion for the establishment of the Analytical Society, who published their memoirs, containing many valuable papers on Analysis, until the opposition somewhat subsided. The members of the Analytical Society were those who have subsequently become the most distinguished philosophers of Great Britain, of whom it is sufficient to particularise a Herschel, Babbage, and Peacock. These gentlemen supported Woodhouse in his endeavours, by the translation of Lacroix, with excellent notes, and a splendid collection of examples in

the Differential and Integral Calculus, in the Calculus of Finite Differences, and in that of Functions. First came Woodhouse's Analytical System of Trigonometry; prior to which the English were ignorant of the whole subject of the Arithmetic of Sines as invented by Euler, the former systems of trigonometry not containing even the theorems $\sin.(A \pm B) = \sin. A . \cos. B \pm \cos. A \sin. B$, and $\cos. (A \pm B) = \cos. A \cos. B \mp \sin. A \sin. B$; then Woodhouse's Treatise on Analytical Calculation; then his Astronomy, which was treated analytically; then the works of the triumvirate already mentioned, and subsequently a number of excellent analytical works on almost every branch of the mathematics. Professor Babbage justly observes in his work 'On the Decline of Science in Great Britain,' "at Cambridge, at least, they have now taken so decided a turn for analysis, that, although still much behind our neighbours on the Continent, it is to be presumed that ere long they will overtake them." Yet, notwithstanding such is a true though brief sketch of the progress of analytical investigation, in the land which produced him who discovered such laws of universal nature as required in their full development the utmost powers of the most refined analysis, that which deserves the highest consideration has hitherto been the most neglected. At Trinity College, however, a few papers at the Annual Examinations in June have been proposed on this subject—at the utmost half-a-dozen. To collect, indeed, examination papers in Analytical Geometry, at the "fountain-head of British science," would be difficult, even to a moderate extent. Still that

subject has taken root, and will henceforth flourish abundantly, the soil being rich and seed sound and prolific.

The works already extant on this important and elegant subject were produced a few years since, when, although at least fifty years old on the Continent, in this country it was in its very infancy.

These works, consequently, display much ingenuity applied to investigations whose results are no longer held *analytically* of any value. In treating of the conic sections, for instance, nearly one-half of them consists of the theory of *conjugate diameters*, and similar topics. Now, although in a geometrical system of conic sections it is necessary to investigate their geometrical properties, as applicable to the establishment of such propositions as occur in Newton's Synthetical view of Natural Philosophy, yet for analytical discussion they are totally useless—mere objects of curiosity. If a person still adhere to geometrical methods, let him prepare himself in geometrical conics; but if he adopt the prodigious powers of Analysis, all the knowledge of the conic sections that he will require for the most elaborate and refined speculations will not be that of their conjugate diameters and other propositions of a like character, but merely their rectangular and polar equations; the equations of their tangents, normals, and circles of curvature. In the whole extent of the five quarto volumes of the Mécanique Céleste, the two quartos of the Mécanique Analytique, the numerous analytical works of Monge and Euler, no reference whatever is made to these geometrical properties of the conic sections. Then why should they still

encumber our elementary treatises? The works alluded to (some two or three in number) do contain this mathematical rubbish; but the fault is not that of the authors themselves, but of the prevailing vicious taste of their British contemporaries. Recently that taste has given way to one more refined; a change principally due to Mr. Whewell, who, by his elegant work on Dynamics, has rendered the chief propositions of Newton's 'Principia' into modern Analysis; thus insensibly, as it were, turning the course of study into its proper channel.

It is in accordance with the views entertained by this enlightened philosopher, and by many others of the University, that, in the present work on Analytical Geometry, the author has endeavoured, in the field he has undertaken to cultivate, to eradicate as many as he can of those weeds of science which have hitherto been suffered to choke the growth of the really valuable crop. In geometry, supplemental chords, diameters, conjugate diameters, &c., may be very interesting, and even necessary to the system, but they are quite alien to useful analysis. All that is there to be considered, with respect to curves, is the discussion of their equations, as it has before been remarked; of those of their tangents, normals, and circles of curvature; and wherefore? Simply because the equations of curves lead to their description; that bodies in motion move in lines either right or curved; that the direction of motion is the tangent of the curve; that the pressure of any force upon a curve is effective along the normal; and that the motion of a body in the circle of curvature is the same as that in the curve at the same point—which are reasons intelligible only to

those who have understood the mixed mathematics; but, being intelligible to those, are good and true reasons.

The work commences with a number of definitions, several of which, it is hoped, will be found improvements on former definitions of the same things, more especially that of an angle. Euclid's definition of an angle is acknowledged to amount to nothing. He says that "a rectilineal angle is the inclination of two right lines to one another, which meet, but are not in the same direction." But what is inclination? Besides, if we adopt the popular acceptation of the term, what are the consequences from this definition? A right line which is at right angles to another has no inclination to it. Consequently, by Euclid's definition, a right angle is not an angle. In this work it is shown that an angle, properly considered, is an *appreciable* ratio of two infinite plane areas. The very basis of all mathematical science being definition, too much care cannot well be bestowed on such preliminaries. Euclid's principal defects are of this nature.

Throughout these pages it has also been considered of importance to adhere as much as possible to one uniform system of notation. It would be of infinite service in the sciences could any unvarying notation be discovered, that should at once be recognised and universally adopted as the best devisable. It is already agreed by all men of science, that the first letters of the alphabet shall denote known or given quantities; that the last shall represent the unknown quantities of equations; that the latter shall also stand for variable quantities, which double use of x, y, z is a defect; that d denotes

the operation of a differential; that e is the base of the Napierian system of logarithms, and also the eccentricity of a conic section—another slight imperfection; that f, standing before a variable, implies a function of that variable; that g is the value of the accelerating force of gravity; h the increment of a variable; that l, standing before a quantity, means the Napierian logarithm of that quantity; that m, in the most approved writers, signifies the mass of a body; that n is an abstract number or ratio, and generally used as a constant index of a power for which p and q are also frequently used; that r is the radius vector of a curve; that s is the length of a curve; that t is the time of a body's motion; that u is the value of a function of a variable; and that v is the velocity of a body, which v is also used in French writers generally, and also in some English authors, to signify an angle in the conic sections called the true (*vrai*) anomaly. Now it appears, from the preceding view of the analytical use that is made of the alphabet, that in many respects that use is already universally conventional. It is of great importance that all symbols of quantity or of operation should be characterized by the same conventional universality. A great degree of the abstruseness of mathematical investigation would be annihilated, were it agreed among mathematicians invariably to adopt the same letter to denote the same species of quantity or operation, distinguishing the different quantities or operations of the same species merely by one, two, three, &c., accents or 'dashes.' A proposition thus conducted would be a picture to the eye, no

less than the '*mind's eye*,' the corporeal being thereby rendered subservient to the intellectual vision.

Much has been said and written on the subject of notation at the university, evincing considerable ingenuity in the proposers of the new symbols of operation; but it appears to the author of these pages, who has had some experience in science, that although symbols may be devised greatly to abbreviate propositions, yet what is gained in the brevity of the proposition will be lost in the symbol; that although a quarter of an hour may be gained in the proposition when the meaning of the symbol is understood, yet that quarter of an hour may be lost in comprehending the power of the symbol. What is gained in the proposition is lost in the symbol.

Professor Airy, in his Tracts, has, however, made one reform in notation that ought generally to be adopted, not because it abridges operations, but because of its being an improvement of the metaphysics of the Integral Calculus. The improvement alluded to is $\int_x fx$ for $\int dx . fx$.

Throughout the present work, the author has endeavoured to preserve the uniform notation here described and recommended; and entertains no doubt that other mathematicians will co-operate with him, or rather with those who have for many years been contributing to accomplish this great desideratum.

With regard to the substance of the work, the reader is referred to a copious detail in the *Contents*. Separate sections are devoted to the Theories of a Point, Right Line, Circle, Parabola, Ellipse, Hyperbola, the General Conic Sections, the Sections of a Cone, Transposition of

Co-ordinates, and the Discussion of Equations of Two Dimensions; thus forming a complete system of Analytical Conic Sections.

In Section XI. a slight notice is taken of the other more useful curves, merely giving their definitions and equations.

Section XII. gives a list of the Equations of such Curves as are illustrative of the Differential Calculus.

And Section XIII. concludes the work with a few of the more interesting General Properties of Curves.

These latter sections are very brief, because of the comparative inutility of the subjects of which they treat. In almost every investigation in natural philosophy the only curve to be considered is some one of the conic sections. The catenary, epicycle, and harmonic curve, are almost the only exceptions, the other curves having now become matters of mere curiosity. The general properties of curves also are, as far as natural philosophy is concerned, objects of unprofitable speculation, although that great analyst, Waring, has thought it worth his valuable time to waste so much thought upon them in his 'Proprietates Algebraicarum Curvarum.' In fact, it has already become manifest that as Analysis is rendered more perfect the consideration of curves will be the less indispensable; and when the methods of approximation in Analysis shall be complete, that conic sections themselves may be dispensed with— at least as far as the motions of bodies both celestial and terrestrial are concerned. These may even then be of use in the theory of the figure or shape of bodies, but of none with regard to their motions. What value do

we attach to the path of a ship on the ocean? Is it not enough that at any instant we can estimate her position as determined by her latitude and longitude? Thus, were it not for the imperfection of the analytical methods of approximation, in which it is convenient for the first stage of the approximation to consider the orbit of a planet, an ellipse of small eccentricity, the nature of the orbit or path of the body might be wholly neglected, it being sufficient to ascertain the position of the moving body at any instant of time, as given by the rectangular or polar co-ordinates of the centre of that body. It is, indeed, a maxim among modern mathematicians that the perfection of Analysis will require the aid of no other lines than those of the co-ordinates.

Much attention has been paid in this work to the symmetrical and homogeneous forms of the equations of lines; whereby the meaning and nature of every letter or symbol of quantity is rendered evident on inspection. For instance, the equation of the right line is always put under the form $\frac{x}{a} + \frac{y}{b} = 1$; from which it appears at sight that a and b are the distances of the intersections of the right line with the axes of x and y from the origin of co-ordinates. Also $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation of the ellipse, is both homogeneous and symmetrical with respect to the co-ordinates and the constants, which constants a and b we recognise as the semi-axes. Many advantages are found to arise from preserving equations in homogeneous form, of which not the least consists in its presenting a continual source of verification; it being

evident that the results of any operations upon homogeneous forms must themselves be homogeneous. Numerous are the instances in which an attention to the homogeneity of expressions has led to the correction of errors even in the present work. But the greatest advantage appears to be that of presenting to the mind the precise nature of every symbol, and the exclusion of any preference or distinction with regard to the *co-ordinates.* By the usual distinction of *abscissa* and *ordinate,* several eminent authors in the differential calculus and other subjects have only treated a subject with reference to the ordinate, whereas similar results would have been obtained had a like investigation been instituted with respect to the abscissa. For these reasons, in this work, the term abscissa is altogether omitted, as being worse than useless. Should, however, any one wish to retain the usual forms it is easy for them to write $y = a\,x + b$ instead of $\frac{x}{a} + \frac{y}{b} = 1$, and $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ for $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and so on. But in the former case it will not be so easy for them to recognise the nature of a and b; and yet it is evident that since in $y = a\,x + b$, y and x are both right lines, a must be an abstract number or ratio, and b a right line. In fact, a is the tangent of an angle or the ratio $\frac{b}{a}$ derived from the homogeneous equation

$$\frac{x}{a} + \frac{y}{b} = 1.$$

The homogeneous and symmetrical forms of equations of curves have been the rather adopted in this treatise,

because the idea has already occurred to some authors both here and abroad.

In geometrical treatises on conic sections is always to be found the investigation of the radius and chords of curvature, conducted, as in the first section of Newton's 'Principia,' by the Method of Ultimate Ratios. Not only the radius and chords of the circle of curvature, but the equation of that circle, as also the equation of the evolute for each conic section, have been here obtained algebraically. The equations of the tangent and normal in each are also found; which equations render the consideration of 'sub-tangents' and 'sub-normals' useless. It is presumed that the method here used for determining the radius, &c. of curvature is a novelty. It is purely algebraical, neither employing Ultimate Ratios, Vanishing Fractions, the Differential Calculus, nor any other theory that depends upon evanescent quantities. The principle consists in first finding the equation of a secant circle passing through any three points (a, b), (a', b'), (a'', b'') of the curve, and eliminating from this equation the fractions $\frac{a - a'}{b - b'}$, $\frac{a - a''}{b - b''}$ or $\frac{a' - a''}{b' - b''}$, and then supposing two of the points to coincide with the third. The same may be effected in a similar manner for any curve whatever; and since all Singular Points depend upon the circle of curvature, the whole application of the differential calculus to the theory of curves may thus be superseded by common algebra.

London, Nov. 15, 1835.

AN

ALGEBRAIC SYSTEM OF CONIC SECTIONS,

AND OF THE

HIGHER GEOMETRY.

DEFINITIONS.

ARTICLE 1. DEFINITION.—A VOLUME or SOLID is any portion of space.

2. DEF.—A SURFACE is the boundary of a volume.

3. DEF.—A LINE is the boundary of a surface.

4. DEF.—A POINT is the extremity of a line.

5. DEF.—A RIGHT LINE is that which can pass through every two, but not through every three or more points in space.

6. DEF.—A CURVE is that line which does not coincide, either wholly or in part, with the straight line joining every two of its points.

7. DEF.—A PLANE is that surface which can pass through every three, but not through every four points in space.

8. DEF.—A CURVED SURFACE is that which does not coincide either wholly or in part with the plane passing through any three of its points.

9. DEF.—A CURVE OF DOUBLE CURVATURE is that whose points are not all in the same plane.

10. Def.—The unit of length, or linear unit, is a right line joining two given points.

The right line thus agreed upon as the standard of linear measure may be an inch, a foot, or any other conventional distance.

11. Def.—The length of a right line is the number of linear units, or parts of a linear unit, which it contains.

12. Def.—The length of a curve is that of a right line, every point of which being supposed to fall upon the curve, the right line and curve wholly coincide.

13. Def.—*An* angle *is the ratio of the plane surface bounded by two infinite right lines which meet, to the plane surface on all sides infinitely extended about the point where they meet.*

Thus the $\angle$ B A C is the ratio of the plane surface bounded by A B, A C extended to ∞ to the unbounded plane of the paper about the point A.

Since the plane surface bounded by A B, A C (Fig. 1.) is infinite as well as the unbounded plane surface about A, an angle, by this definition, is the ratio of two infinites. But it is well known, and can be explained in a familiar manner, that the ratio of two infinites is nothing more than a 'vanishing fraction,' which may be nothing, finite or infinite, according to its form.

Analogous to this definition of an angle are the definitions of the trigonometrical functions of an angle ; as of the *sine, tangent, secant,* &c.

14. Def.—A right angle is either of those angles which one right line makes with another on the same

side of it, when, supposing one of them to be applied to the other, they coincide exactly.

15. Def.—The ANGULAR UNIT is a given part of a right angle.

> Thus the 90th part, or a degree; the 90 × 60th part, or a minute, &c.; may be taken for the angular unit.

16. Def.—ANGULAR LENGTH, or ANGULAR DISTANCE, is the number of angular units or parts of a unit, which the angle contains.

17. Def.—RECTANGULAR CO-ORDINATE AXES are two right lines of indefinite length making four right angles at their common point.

> This common point is called THE ORIGIN OF CO-ORDINATES.

18. Def.—OBLIQUE CO-ORDINATE AXES are two right lines of indefinite length making four angles not right angles at their common point.

19. Def.—The CO-ORDINATES OF A POINT in the plane of the co-ordinate axes are the right lines which are drawn from that point parallel to the axes, and terminated by them and the point.

20. Def.—The POLAR CO-ORDINATES OF A POINT in the plane of the co-ordinate axes are the straight line drawn from it to the origin of co-ordinates, which is also called THE RADIUS VECTOR; and the angle between the radius vector and a co-ordinate axis, which angle is called THE TRACED ANGLE.

Def.—If a point be in the angle Y O X, (Fig. 2.) O being the origin, and X X′, Y Y′ the axes of co-ordinates,

the co-ordinates O M, P M are considered positive; if in Y O X′, then O M′ is negative and M′ P′ positive; if in X′ O Y′, then both are called negative; and if in X O Y′, then O M is positive and M P‴ is negative.

Conversely. If the co-ordinates be positive, they are to be measured along O X, O Y, from the origin O; but in the contrary direction when negative.

DEF.—The DISTANCE between any two points is the length of the right line which joins them.

DEF.—A CO-ORDINATE PLANE is the plane which passes through the origin and one point in each axis of co-ordinates.

NOTATION.

21.—1. All abstract quantities, that is, numbers either integer or fractional, or ratios, are denoted by

$$n \text{ or } N.$$

2. The co-ordinates of a point in the plane of co-ordinate axes, when that point is, and therefore its co-ordinates are given, by

$$a, b.$$

3. The co-ordinates of a point in the plane of co-ordinate axes, when its position is, and therefore its co-ordinates are, variable, by

$$x, y.$$

4. The polar co-ordinates of a point in the plane of co-ordinate axes, when the point is given, by

$$R, \alpha;$$

R being the radius vector and α the traced angle; but when its position is variable, by

$$r, \theta.$$

5. "*The point* (a, b) or (x, y)" means the point whose co-ordinates are a, b, or that at which they are x, y, respectively.

Thus the point (1, 2) means that point the lengths of whose co-ordinates are one and two linear units.

6. "*The point* (R, α) or (r, θ)" means the point whose polar co-ordinates are R, α, or r, θ, respectively.

7. "*The angle* (S, S′) or (s, s')" means that which is contained by the right lines S, S′, or by s, s' respectively.

Section I.

THEORY OF A POINT IN THE PLANE OF CO-ORDINATES.

It may be premised that the use of co-ordinate axes in determining the position of points, and thence in the general theory of curves, and consequently of those which are the orbits or paths of all moving bodies (and there is nothing in the universe which does not move), is of the utmost importance to science. The idea, a most happy one, first occurred to Maclaurin, who, in his elaborate Geometrical Exposition of Fluxions, gave us the desideratum, but in so rough a state, that he himself was quite unconscious of the value of what has subsequently led the way to such a state of mathematical knowledge that it may be considered as verging on perfection.

Relatively to us, the sun is more stationary than any other object which we are permitted to contemplate; excepting, perhaps, what are unscientifically termed the fixed stars. Be that as it may, as the sun most imme-

diately attracts all his dependents towards him, and effects it without any great disturbance to his own repose, we naturally regard him as the most stationary. Hence, with respect to the solar system, and all the operations that belong to it, it is proper and sufficient in all astronomical and other physical questions, all of which depend upon motion, to make him the *origin of co-ordinates.*

Now, the earth, as even those who have only learnt *Popular Astronomy* must know, compelled by his attraction, moves in an orbit about the sun, and at all times does not considerably deviate from the same plane. The earth we may suppose a point, which, compared with the sun, she almost is; and this point, moving in the ecliptic, will every instant have a different position with respect to the sun. From which it is clear that it is important to have some means of ascertaining the position of a body always moving in the same plane with reference to some known point or body, stationary with respect to the system. This is effected in two ways, by ascertaining the rectangular or polar co-ordinates of the moving body at the given instant; the plane in which the body moves being that of the co-ördinate axes, and the origin of co-ordinates, the quiescent body.

Thus much has been said, by way of introduction, not for the purpose of making the subject clearer, but to impress upon the mind of the student, that, when he enters upon Analytical Geometry, he treads a soil the richest perhaps, in its products, of all the lands of pure science.

22. PROPOSITION.—*Given* (a, b) *the rectangular co-ordinates of a point in a given plane to find the point.* (Fig. 3.)

If a and b be both positive, measure along O X, O Y, as many linear units as are contained in a and b respectively. Let these lengths be O A, O B. Complete the parallelogram O P. Then P is the point required.

For ∵ P A, P B are parallel to the axes of co-ordinates, they are the co-ordinates of the point P; and $PB = AO = a$, $PA = OB = b$. ∴ P is the point (a, b).

If a be negative and b positive; that is, if $(-a, b)$ be the point, measure a along O X′, and completing the ▭ O P′, P′ will be the point required.

If a and b be both negative; that is, if $(-a, -b)$ be the point, measure a along O X′, and b along O Y′, and completing the ▭ O P″, P″ will be the point required.

If the point be $(a, -b)$; measure a along O X, and b along O Y′, and completing the ▭ O P‴, P‴ will be the point required.

The demonstration of the first case applies also to the three last.

If the co-ordinate axes be oblique, the same construction applies exactly.

EXAMPLES.

Ex. 1.—*To find the point* (1, 2). (Fig. 3.)

Take $OA = 1$, $OB = 2$. Complete the ▭ O P. Then P is the point required.

Ex. 2.—*To find the point* (− 3, 2). (Fig. 4.)

Take O A' = 3, O B = 2. Then completing the ▭, P′ will be the point.

Ex. 3.—*To find the point* (—2, —1). (Fig. 4.)

The point is in the angle X′ O Y′ as P″.

Ex. 4.—*To find the point* (3, – 2). (Fig. 4.)

This point is in the angle X O Y′ as P‴.

Ex. 5. *To find the points* (2, 4), *the oblique axes being inclined at an angle of* 45°. (Fig. 5.)

The construction is evident from the diagram.

23. Prop.—*To find the distance* S *between two points* $(a, b), (a' b')$. (Fig. 6.)

Find the points and let them be P, P′; then joining P P′, and drawing P′ B ∥ O X to meet P A in B, we have

$$(PP')^2 = (PB)^2 + (BP')^2$$

$$\text{or, } S^2 = (b - b')^2 + (a - a')^2$$

$$\therefore S = \sqrt{\{(a - a')^2 + (b - b')^2\}}$$

Ex. The distance between the points (2, 3,) and (– 2, – 3), is $2\sqrt{13}$.

24. Prop.—*To find the distance* S *between two points* (a, b), (a', b'), *the angle between the axes being* α. (Fig. 7.)

Let P, P′ be the points, ∠ Y O X = α. Draw P B parallel to O Y. Then since ∠ P B P′ = $\pi - \alpha$,

$$\cos.(\pi - \alpha) = \frac{PB^2 + P'B^2 - PP'^2}{2\,PB \times P'B}$$

$$\therefore -\cos.\alpha = \frac{(b - b')^2 + (a - a')^2 - S^2}{2\,(b - b')\,(a - a')}$$

$$\therefore S^2 = (a - a')^2 + (b - b')^2 + 2\,(a - a')\,(b - b') \times \cos.\alpha.$$

$\therefore S = \sqrt{\{(a - a')^2 + (b - b')^2 + 2(a - a') \times (b - b') \cos. \alpha\}}$.

Which, when $\alpha = 90°$, coincides with the value in Prop. 2.

Ex. The distance between the points (3, 2), (– 4, 6), when the axes make an $\angle$ of 60°, is $\sqrt{37}$.

25. Prop.—*To find the point whose polar co-ordinates are* R *and* α. (Fig. 8.)

At the origin O, make the angle X O A = α, and measure O P along O A = R. Then P is the point required.

For the co-ordinates of P are R and α.

Ex. Required the point (6, 50°).

Make X O A = 50°, and along O A measure O P six linear units. Then P will be the point required.

26. Prop.—*To find the distance* S *between the points* (R, α), (R′, α'). (Fig. 9.)

Let P, P′ be the points; then X O P = α, X O P′ = α', O P = R, O P′ = R′ and

$$\cos. POP' = \frac{(OP)^2 + (OP')^2 - (PP')^2}{2 OP \times OP'}$$

$$\text{or,} \qquad \cos. (\alpha - \alpha') = \frac{R^2 + R'^2 - S^2}{2 R . R'}$$

$$\therefore S^2 = R^2 + R'^2 - 2 R . R' . \cos. (\alpha - \alpha')$$

$$\therefore S = \sqrt{\{R^2 + R'^2 - 2 R . R' . \cos. (\alpha - \alpha')\}}.$$

Ex. The distance between the points (5, 60°), and (4, 45°) is

$$\sqrt{(41 - 40 \cos. 15°)}$$

$$\text{or } \sqrt{\{41 - 20\sqrt{(2 + \sqrt{3})}\}}$$

$$\text{or } \sqrt{\{41 - 10\sqrt{2}.(\sqrt{3} + 1)\}}$$
$$\text{or } \sqrt{\{41 - 10(\sqrt{6} + \sqrt{2})\}}$$

which may easily be reduced to a decimal form and approximate value.

SECTION II.

THEORY OF A RIGHT LINE IN A CO-ORDINATE PLANE.

27. DEF.—The EQUATION OF A LINE is that which expresses the relation between the co-ordinates of any point in that line, and it is called RECTANGULAR, OBLIQUE, or POLAR, according as the axes are rectangular, oblique, or polar.

From the Definition of a Line, that is, from certain of its properties, it is a problem of no difficulty to find its equation; and conversely from the equation, the properties of the curve may easily be discovered. These observations will be fully exemplified in the subsequent pages.

28. PROP.—*From the definition of a right line to find its rectangular equation.*

Since a right line passes through two points, let those points be two given ones (Fig. 10), A, A′, viz. (a, b), (a', b'); and suppose P any other point (x, y); that is, let

$$\left.\begin{matrix} OB = a \\ AB = b \end{matrix}\right\} \left.\begin{matrix} OB' = a' \\ A'B' = b' \end{matrix}\right\} \left.\begin{matrix} OM = x \\ PM = y \end{matrix}\right\}$$

and draw AN ∥ OX meeting A′B′, PM in C and N; then by similar △s, ACA′, PNA, we have

$$\text{PN} : \text{A}'\text{C} :: \text{AN} : \text{AC}$$

$$\text{or } y \sim b : b' \sim b :: x \sim a : a' \sim a$$

$$\therefore (a \sim a')(y \sim b) = (b \sim b')(x \sim a)$$

$$\therefore (b \sim b')\,x + (a \sim a')\,y = b\,(a \sim a') \sim a\,(b \sim b')$$
$$= a\,b' \sim a'\,b$$

$$\therefore \frac{x}{\left(\dfrac{a\,b' \sim a'\,b}{b \sim b'}\right)} + \frac{y}{\left(\dfrac{a\,b' \sim a'\,b}{a \sim a'}\right)} = 1$$

which is the equation to the right line passing through the points (a, b), (a', b'), in its most symmetrical and simple form.

29. Cor. 1.—Make $\dfrac{a\,b' \sim a'\,b}{b \sim b'} = s$ and $\dfrac{ab' \sim a'b}{a \sim a'} = s'$

$$\text{Then} \quad \frac{x}{s} + \frac{y}{s'} = 1$$

is the equation in its proper homogeneous form, s and s' being certain straight lines, which we shall subsequently construct.

30. Cor. 2.—Let the given points A, A′, be taken, the former in the axis of x, and the other in that of y; then $b = o$, and $a' = o$, and

$$\frac{x}{a} + \frac{y}{b'} = 1$$

is the equation to the straight line which passes through the points (a, o), (o, b').

Hence then, generally, the equation of the right line which passes through the points (a, o), (o, b), is

$$\frac{x}{a} + \frac{y}{b} = 1. \ldots\ldots (2)$$

which is the form in which we shall always consider the equation of a right line.

31. Cor. 3.—If D and E (Fig. 10.) be the points where the right line cuts the axes of x and y; then

$$\text{OD} = \frac{ab' \sim a'b}{b \sim b'} \text{ and } \text{OE} = \frac{ab' \sim a'b}{a \sim a'}$$

which lines are obviously the geometrical construction of the algebraic expressions.

Examples.

Ex. 1.—*Required the equation of the right line which passes through the points* (1, 2), (2,3).

Assume the equation of the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

in which a and b are to be determined; then, since the line passes through the points (1, 2), (2, 3), we have

$$\left.\begin{aligned} \frac{1}{a} + \frac{2}{b} &= 1 \\ \frac{2}{a} + \frac{3}{b} &= 1 \end{aligned}\right\}$$

$$\therefore \quad \frac{3}{a} + \frac{6}{b} = 3$$

$$\frac{4}{a} + \frac{6}{b} = 2$$

$$\therefore \quad \frac{1}{a} = -1 \qquad \therefore \; a = -1$$

$$\text{and } \frac{2}{b} = 1 - \frac{1}{a} = 2$$

$$\therefore \quad b = 1$$

$\therefore$ the equation required is $-x + y = 1$.

Ex. 2.—*The equation of the right line passing through* $(-4, 5)$ $(2, 6)$ *is*

$$\frac{x}{-34} + \frac{y}{\left(\frac{17}{3}\right)} = 1$$

Ex. 3.—*The equation of the right line passing through the points* $(-3, 5)$ $(-4, 5)$ *is*

$$y = 5.$$

Ex. 4, &c.—*The equation of the right line passing through the points* $(2, 3)$ $(5, 4)$ *is*

$$\frac{x}{-7} + \frac{3y}{7} = 1$$

of that through

$(-2, 3)$, $(5, 4)$ is $\frac{-x}{23} + \frac{7y}{23} = 1$

$(2, -3)$, $(5, 4)$ is $\frac{7}{23}x - \frac{3}{23}y = 1$

$(2, 3)$, $(-5, 4)$ is $\frac{1}{23}x + \frac{7}{23}y = 1$

$(2, 3)$, $(5, -4)$ is $\frac{7}{23}x + \frac{3}{23}y = 1$

$(-2, -3)$, $(5, 4)$ is $x - y = 1$

$(-2, 3)$, $(-5, 4)$ is $\frac{1}{7}x + \frac{3}{7}y = 1$

$(-2, 3)$, $(5, -4)$ is $x + y = 1$

$(2, 3)$, $(-5, 4)$ is $-x - y = 1$

$(2, -3)$, $(5, -4)$ is $-\frac{x}{7} - \frac{3}{7}y = 1$

$(2, 3), (-5, -4)$ is $-. x + y = 1$

$(-2, -3), (-5, 4)$ is $-\frac{7}{23} x - \frac{3}{23} y = 1$

$(-2, -3), (5, -4)$ is $-\frac{1}{23} x - \frac{7}{23} y = 1$

$(-2, 3), (-5, -4)$ is $-\frac{7}{23} x + \frac{3}{23} y = 1$

$(2, -3), (-5, -4)$ is $\frac{x}{23} - \frac{7}{32} y = 1$

$(-2, -3), (-5, -4)$ is $\frac{x}{7} - \frac{3}{7} y = 1$

Singular Cases.

If the points be in the axes of co-ordinates; that is, if the points be (a, o), (o, b); then the equation is obtained without any process, by assuming it at once to be

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Thus, the equation passing through the points

$$(2, o), \ (o, 3\tfrac{1}{2}),$$

is $$\frac{x}{2} + \frac{y}{3\frac{1}{2}} = 1.$$

If the points be (o, o), (a', b'), then the right line passes through the origin and in the assumed equation

$$\frac{x}{a} + \frac{y}{b} = 1.$$

both a and $b = 0$.

$\therefore$ the equation is of the form

$$x + y \cdot \frac{a}{b} = a = 0,$$

and $$a' + b'\frac{a}{b} = 0;$$

$$\therefore \quad \frac{a}{b} = -\frac{a'}{b'};$$

$\therefore$ the equation in this case is

$$x - \frac{a'}{b'}y = 0;$$

or the equation of the right line passing through the origin and the point (a, b) is

$$y = \frac{b}{a}x \;\ldots\ldots\ldots\; (3).$$

If the right line pass through the points (o, o), (o, b), its equation is

$$y = \frac{b}{o} \times x;$$

which signifies that in order that b and y be finite, x must be always zero; that is, the right line is the axis of y.

In like manner, if the line passes through the points (o, o), (a, o), its equation is

$$y = \frac{o}{a} \times x;$$

or $$x = \frac{a}{o} \times y;$$

$\therefore$ for x and a to be finite, it is necessary that y shall always be zero; that is, the line is the axis of x.

If the line pass through the points (a, b), (a, b'), then assuming its equation to be

$$\frac{x}{a''} + \frac{y}{b''} = 1;$$

we have
$$\left.\begin{aligned}\frac{a}{a''} + \frac{b}{b''} &= 1\\ \frac{a}{a''} + \frac{b'}{b''} &= 1\end{aligned}\right\}$$

$$\therefore \quad \frac{1}{b''}(b - b') = 0; \qquad \therefore \quad \frac{1}{b''} = 0;$$

$$\therefore \quad \frac{a}{a''} = 1; \qquad \therefore \quad a'' = a;$$

$\therefore$ the equation is

$$x = a \text{ .... } (4).$$

Similarly, if it pass through the points

$$(a, b,) \ (a', b);$$

then the equation is

$$y = b \text{ .... } (5).$$

and in these cases, it is clear that the lines are respectively parallel to the axes of y and x.

RECAPITULATION.

32. 1. The equation of the right line whose points are

$$(a, b), \ (a', b'),$$

is
$$\frac{x}{\left(\frac{ab' \sim a'b}{b \sim b'}\right)} + \frac{y}{\left(\frac{ab' \sim a'b}{a \sim a'}\right)} = 1.$$

2. The equation of that two of whose points are

$$(a, o), \ (o, b),$$

is $\dfrac{x}{a} + \dfrac{y}{b} = 1.$

3. Of that which passes through the origin and the point (a, b) is

$$y = \frac{b}{a} x.$$

4. The equations of those which pass through (o, o), (o, b), and through (o, o), (a, o), are respectively

$$y = y, \text{ and } x = x,$$

which are indefinite expressions, and may be of any magnitude whatsoever.

5. Of those which pass through (a, b), (a, b'), and through (a, b), are respectively

$$x = a, \ y = b.$$

It will be sufficient that the student commit to memory the simple formula

$$\frac{x}{a} + \frac{y}{b} = 1,$$

as that to which the equation of every conceivable right line can be reduced.

From the preceding results, it is evident that the equation to every right line is a simple indeterminate equation of the form

$$\frac{x}{a} + \frac{y}{b} = 1;$$

in which (a, o), (o, b) are the points in which the right line meets the axes of x and y respectively.

It may here be observed, that in the equations of lines we first employ *Variable Quantities.*

33. DEF.—*Variable Quantities,* or *Variables,* are those which, between any finite limits, admit an indefinite number of values.

Thus, in the equation of a right line, viz.,

$$\frac{x}{a} + \frac{y}{b} = 1,$$

whatever we assume the value of x, from $-\infty$ to o, and

thence to $+\infty$, a corresponding value of y can always be found. Hence x and y, having an infinite number of corresponding values, are variables.

If we have as many equations as unknown quantities, the corresponding values of the unknown quantities are of limited number, and are consequently not variable.

Also the pairs of values in one equation, involving two unknown quantities, which arises from an indeterminate problem, although they may be infinite in number, yet as from the conditions of the problem that number is not infinite between every finite limit, equations so circumscribed by conditions do not contain variables as unknown quantities. But there being an infinite number of points in any finite portion of a line, throughout that portion there is an infinite number of pairs of co-ordinates of such points, and consequently the co-ordinates of the general point of even a portion of a curve are variables.

It is thus we first obtain a notion of variables, and hence it is that we commence a preparation for the Differential Calculus or Fluxions; a calculus which is wholly distinct from Algebra, employing indeed the rules of Arithmetic, Elementary Geometry, and Algebra, but passing to the consideration of Variable Quantities; that is, to speak in popular language, to things which grow or decay.

Hence it will be useful to bear in mind that the things which are represented by x, y in this subject are very different in their nature from what are denoted by the same symbols in common Algebra.

34. Prop.—*Given the rectangular equation of a right line to construct it; that is, to find any two points through which it passes.*

Remark.—It is generally most easy, in constructing right lines, to find the points in the axes through which they pass, and this method will be generally adopted; but it is equally right to find any two other points. This being premised, let

$$N x + N' y = s$$

be the given equation, N and N′ being abstract numbers, and s the length of a right line. This equation is reducible to

$$\frac{x}{\left(\frac{s}{N}\right)} + \frac{y}{\left(\frac{s}{N'}\right)} = 1.$$

Hence the points in which the line and the axes intersect are

$$\left(\frac{s}{N}, o\right), \left(o, \frac{s}{N'}\right).$$

Otherwise,

Let $y = 0$; then $x = \frac{s}{N}$.

Let $x = 0$; then $y = \frac{s}{N'}$.

Hence two points in the right line required are $\left(\frac{s}{N}, o\right)$ and $\left(o, \frac{s}{N'}\right)$; and if they be joined, and the line joining them indefinitely produced, it will be the line of which the equation is

$$N x + N' y = s.$$

EXAMPLES.

Ex. 1.—*To find the right line whose equation is*

$$5x + 6y = 7.$$

First we get

$$\frac{x}{\left(\frac{7}{5}\right)} + \frac{y}{\left(\frac{7}{6}\right)} = 1$$

$\therefore$ the points in which the line cuts the axes of x and y are respectively

$$\left(\frac{7}{5}, o\right), \left(o, \frac{7}{6}\right).$$

And if a right line be drawn through these points, it will be the one required.

Otherwise,

Let $y = 0$; then $x = \frac{7}{5}$; $\therefore$ one point is $\left(\frac{7}{5}, o\right)$

Let $x = 0$; then $y = \frac{7}{6}$; $\therefore$ another point is $\left(o, \frac{7}{6}\right)$

which points are the same as before.

Ex. 3.—*To construct the right line whose equation is*

$$\frac{2}{3}x - \frac{3}{2}y = 7.$$

Since the equation is transformable to

$$\frac{x}{\left(\frac{21}{2}\right)} + \frac{y}{\left(-\frac{14}{3}\right)} = 1$$

$\therefore$ the points in the axes of x and y through which it passes are

$$\left(\frac{21}{2}, o\right) \text{ and } \left(o, -\frac{14}{3}\right).$$

If O A be measured $= \frac{21}{2}$ linear units, and O B $= \frac{14}{3}$ units, the right line joining A B will be that required. (Fig. 11.)

Ex. 4.—*To construct the right line whose equation is*

$$-5x + 7y = 6.$$

It passes through the points

$$\left(-\frac{6}{5}, o\right), \left(o, \frac{6}{7}\right),$$

and is therefore easily drawn.

Ex. 5.—*To find the right line whose equation is*

$$-9x - 7y = 5.$$

Two of its points are

$$\left(-\frac{5}{9}, o\right) \text{ and } \left(o, -\frac{5}{7}\right).$$

Ex. 6.—*To construct the line whose equation is* $y = Nx$.

$$\text{Since } y - Nx = 0$$

the equation is not of the general form

$$\frac{x}{a} + \frac{y}{b} = 1,$$

and becomes indefinite; but still by finding one point which is not in either axis, that together with the origin, which is one point of the right line, it will be fully determined.

$$\text{Thus let } x = 1; \text{ then } y = N,$$

and two points of the right line are

$$(o, o), (1, N).$$

Ex. 7.—*To construct the lines*

$$\frac{y}{2} = 1, \frac{x}{3} = 1.$$

(1.) In the axis of y measure two units from the origin, and thence draw a line parallel to the axis of x.

(2.) In that of x draw a line parallel to the axis of y, at the distance 3 units from that axis. These are the lines required.

35. Prop.—*To find the polar equation of a right line passing through two given points* (R, α), (R′, α'). (Fig. 12.)

Let A, A′ be the given points, and P any other point whatever (r, θ) in the right line. Then joining P, A, A′ with the origin of co-ordinates, we have

$$\text{O A} = \text{R}, \text{O A}' = \text{R}', \text{O P} = r$$

$$\angle \text{ A O X} = \alpha, \text{A}'\text{O X} = \alpha', \text{X O P} = \theta:$$

$$\left.\begin{array}{l}\text{Also A P} : \text{O P} :: \sin. \text{A O P} : \sin. \text{O A P} \\ \text{and A}'\text{P} : \text{O P} :: \sin. \text{A}'\text{O P} : \sin. \text{O A}'\text{P}\end{array}\right\}$$

$$\left.\begin{array}{l}\text{That is, A P} : r :: \sin. (\alpha \sim \theta) : \sin. \text{O A P} \\ \text{and A}'\text{A} + \text{A P} : r :: \sin. (\alpha' \sim \theta) : \sin. \text{O A}'\text{P}\end{array}\right\}$$

$$\therefore \text{A A}' + \text{A P} = \frac{r \sin. (\alpha' \sim \theta)}{\sin. \text{O A}'\text{P}.}$$

$$\text{and A P} = \frac{r \sin. (\alpha \sim \theta)}{\sin. \text{O A P}}$$

$$\therefore \text{A A}' = \frac{r \sin. (\alpha' \sim \theta)}{\sin. \text{O A}'\text{P}} - \frac{r \sin. (\alpha \sim \theta)}{\sin. \text{O A P}}$$

$$\left.\begin{array}{l}\text{But } \sin. \text{O A P} : \sin. \text{A O A}' :: \text{R}' : \text{A A}' \\ \text{and } \sin. \text{O A}'\text{P} : \sin. \text{A O A}' :: \text{R} : \text{A A}'\end{array}\right\}$$

$$\therefore \frac{1}{\sin. \text{O A P}} = \frac{\text{A A}'}{\text{R}'. \sin. (\alpha' \sim \alpha)} \quad \text{and}$$

$$\frac{1}{\sin. \text{O A}'\text{P}} = \frac{\text{A A}'}{\text{R} \sin. (\alpha' \sim \alpha)}$$

$\therefore$ substituting, we get

$$A\,A' = \frac{A\,A'.\,r.\,\sin.\,(\alpha' \sim \theta)}{R\,.\,\sin.\,(\alpha' \sim \alpha)} - \frac{A\,A'.\,r\,\sin.\,\alpha \sim \theta)}{R'\,\sin.\,(\alpha' \sim \alpha)}$$

$$\therefore\ R\,.\,R'.\,\sin.\,(\alpha' \sim \alpha) = R\,.\,r\,\sin.\,(\alpha' \sim \theta) - R\,.\,r \times \sin.\,(\alpha \sim \theta).$$

Hence, not considering the signs of the terms which depend upon the magnitudes of the traced angles α, α' and θ, the general form of the polar equation to a right line is

$$\frac{r\,.\,\sin.\,(\alpha \sim \theta)}{R'\,\sin.\,(\alpha' \sim \alpha)} + \frac{r\,\sin.\,(\alpha' \sim \theta)}{R\,\sin.\,(\alpha' \sim \alpha)} = 1 \ldots\ldots\ldots (6)$$

which according to other notation is

$$\frac{r\,.\,\sin.\,(r,\,R)}{R'.\,\sin.\,(R,\,R')} + \frac{r.\,\sin.\,(r\,R')}{R\,.\sin.\,(R,\,R')} = 1 \ \ldots\ldots\ (7)$$

that is, the polar equation to a right line passing through two given points is of this form, viz. :—

36. THEOREM.—*The product of the variable vector and the sine of the angle between it and one of the given vectors, divided by the product of the other given vector, and the sine of the angle between the given vectors; plus the product of the variable vector, and the sine of the angle between it and that other given vector, divided by the product of the first given vector, and the sine of the angle between the two given vectors, all together equal unit.*

EXAMPLES.

1. *To find the polar equation of the right line passing through the points* (2, 30°), (3, 45°).

It is clearly

$$\frac{r \sin (30^\circ - \theta)}{3 \, . \sin. 15^\circ} + \frac{r \sin. (45^\circ - \theta)}{2 \sin. 15^\circ} = 1$$

which does not admit of material reduction.

2. *To find the polar equation of the right line passing through the points* (R, *o*), $\left(R', \frac{\pi}{2}\right)$ *or that in which the given points are in the rectangular axes.*

In this case

$$\frac{r \sin. \theta}{R' \sin. \frac{\pi}{2}} + \frac{r \sin. \left(\frac{\pi}{2} - \theta\right)}{R \sin. \frac{\pi}{2}} = 1$$

or,

$$\frac{r \sin. \theta}{R'} + \frac{r \cos. \theta}{R} = 1.$$

This is precisely the result that would ensue from the following problem:

Given the rectangular equation of a right line to find its polar equation.

Let its given points be (a, o), (o, b); then its rectangular equation is

$$\frac{x}{a} + \frac{y}{b} = 1$$

and it is evident from fig. 12, that

$$x = r \cos. \theta, \; y = r \sin. \theta.$$

$$\therefore \frac{r \cos. \theta}{a} + \frac{r \sin. \theta}{b} = 1 \; . \; . \; . \; . \; . \; . \; (8)$$

which may be thus described :—

36. THEOREM.—*The general polar equation of a right line is the product of the variable vector and cosine*

of the traced angle, divided by that vector which falls upon the axis of x, *plus the product of the variable vector, and sine of the traced angle, divided by that vector which falls upon the axis of* y, *all together equal to unit.*

This is the form of the equation which it is best to adopt.

It might have been deduced from the equation $\frac{x}{a} + \frac{y}{b} = 1$; and thus obtained, the process would have been more simple than the investigation of Art. 35. But it is best to accustom students to direct and independent investigations. Every proof should be made to rest as much as possible on definition.

EXAMPLES OF THE GENERAL POLAR EQUATION.

Ex. 1.—*To find the equation of the right line passing through the points* (R, *o*), $\left(\text{R}, \frac{\pi}{2}\right)$.

Assume the equation to be

$$\frac{r \cos. \theta}{a} + \frac{r \sin. \theta}{b} = 1$$

then $$\frac{\text{R}}{a} = 1 \therefore a = \text{R}$$

and $$\frac{\text{R}'}{b} = 1 \therefore b = \text{R}'$$

$\therefore$ the equation required is

$$\frac{r \cos. \theta}{\text{R}} + \frac{r \sin. \theta}{\text{R}'} = 1.$$

Ex. 2.—*To find the equation of the right line passing through the points* (2, *o*), $\left(3, \frac{\pi}{2}\right)$.

the equation is

$$\frac{r \cos.\theta}{2} + \frac{r \sin.\theta}{3} = 1.$$

Ex. 3.—*To find the equation of the right line passing through the points* (1, 2), (– 2, 10), (rectangular co-ordinates).

First, the rectangular equation being assumed

$$\frac{x}{a} + \frac{y}{b} = 1;$$

we have

$$\left.\begin{aligned} \frac{1}{a} + \frac{2}{b} &= 1 \\ -\frac{2}{a} + \frac{10}{b} &= 1 \end{aligned}\right\}$$

$$\therefore \frac{2}{a} + \frac{4}{b} = 2$$

$$\therefore \frac{14}{b} = 3 \quad \therefore \frac{1}{b} = \frac{3}{14}$$

$$\text{Whence} \quad \frac{1}{a} = \frac{4}{7}$$

$$\therefore \quad \frac{x}{\left(\frac{7}{4}\right)} + \frac{y}{\left(\frac{14}{3}\right)} = 1.$$

$\therefore$ the equation required is

$$\frac{r \cos.\theta}{\left(\frac{7}{4}\right)} + \frac{r \sin.\theta}{\left(\frac{14}{3}\right)} = 1.$$

37. Prop.—*Every equation of the form*

$$C\, r \cos.\theta + C'\, r \sin.\theta + C'' = 0,$$

is the polar equation of a right line.

For dividing it by $(-\text{C}'')$ and transposing

$$-\frac{\text{C}}{\text{C}''}\, r \cos.\,\theta - \frac{\text{C}'}{\text{C}''}\, r \sin.\,\theta = 1\,,$$

$$\therefore \quad \frac{r\cos.\,\theta}{\left(-\dfrac{\text{C}''}{\text{C}}\right)} + \frac{r\sin.\,\theta}{\left(-\dfrac{\text{C}''}{\text{C}'}\right)} = 1$$

which is the equation of a right line passing through the points

$$\left(-\frac{\text{C}''}{\text{C}}, o\right), \left(-\frac{\text{C}''}{\text{C}'}, \frac{\pi}{2}\right).$$

Examples.

Ex. 1.—Construct the right line whose polar equation is

$$r \sin.\,\theta = 2.$$

Assume the equation to be

$$\frac{r\cos.\,\theta}{a} + \frac{r\sin.\,\theta}{b} = 1$$

Then $\therefore$
$$\frac{r\cos.\,\theta}{\dfrac{1}{0}} + \frac{r\sin.\,\theta}{2} = 1$$

$\therefore \quad a = \infty$ and $b = 2$, or the right line passes through the points

$$(\infty\,, o), \left(2, \frac{\pi}{2}\right)$$

that is, it is parallel to the axis of x and distant from it by 2 units.

Otherwise. Let $\theta = 0$; then $r = \infty$,

$\therefore$ it is parallel to the axis of x.

$$\text{Let } \theta = \frac{\pi}{2}\text{; then}$$

$$r = 2. \quad \therefore \text{ \&c. \&c.}$$

Ex. 2.—Construct the right line whose equation is $y \cos. x = 3$.

It is a polar equation whose vector is y, and traced angle x; and may $\therefore$ be considered $r \cos. \theta = 3$.

$$\text{Let } \theta = 0, \text{ then } r = 3.$$

$$\text{Let } \theta = \frac{\pi}{2}, \text{ then } r = \infty.$$

$\therefore$ the right line cuts the axis of x at the distance 3 from the origin, and is parallel to the axis of y.

Ex. 3. *To construct the right line whose equation is*

$$2\, r \cos. \theta - 3\, r \sin. \theta = 7.$$

$$\text{Let } \theta = 0, \frac{\pi}{2},$$

$$\text{then } r = \frac{7}{2}, -\frac{7}{3} \text{ respectively.}$$

$\therefore$ the right line passes through the points

$$\left(\frac{7}{2}, o\right), \left(-\frac{7}{3}, \frac{\pi}{2}\right)$$

Recapitulation of the Equations of a Right Line.

$$\left.\begin{array}{l} \dfrac{x}{a} + \dfrac{y}{b} = 1 \\[2ex] \dfrac{x \cos. \theta}{a} + \dfrac{r \sin. \theta}{b} = 1 \end{array}\right\}$$

These are all that are necessary for the theory of the right line, whether it be conducted by aid of rectangular or polar co-ordinates.

Problems on the Right Line.

38. Prob.—*To find the angle which a right line makes with either axis.*

Let the equation be

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \text{or} \quad \frac{r \cos. \theta}{a} + \frac{r \sin. \theta}{b} = 1$$

then, if (s, x) denote the angle between the right line and the axis of x, we have

$$O B = O A . \tan. (s, x)$$

$$\text{But} \quad O A = a, O B = b$$

$$\therefore \quad \tan. (s, x) = \frac{b}{a}$$

$$\therefore \quad \angle (s, x) = \tan.^{-1} \frac{b}{a}$$

Hence $\angle (s, y) = \frac{\pi}{2} - \tan.^{-1} \frac{b}{a} = \tan.^{-1} \frac{a}{b}$.

Hence when s is parallel to the axis of x

$$\frac{b}{a} = 0; \text{ that is, } b = 0 \text{ or } a = \infty.$$

When parallel to that of y

$$\frac{a}{b} = 0; \text{ that is, } a = 0 \text{ or } b = \infty.$$

The theorem arising from the above problem is

39. Theorem.—*The angle between a straight line and either co-ordinate axis is that whose tangent is the ratio of its common ordinates with the axes* y *and* x *respectively.*

40. Prob.—*Given two points (a', b'), (a'', b'') in a right line to find its angles with the axes.*

Assume the equation to be

$$\frac{x}{a} + \frac{y}{b} = 1,$$

then $$\left.\begin{array}{l} \dfrac{a'}{a} + \dfrac{b'}{b} = 1 \\ \dfrac{a''}{a} + \dfrac{b''}{b} = 1 \end{array}\right\}$$

Whence $$a = \frac{a' b'' - a'' b'}{b'' - b'},$$

$$b = \frac{a' b'' - a'' b'}{a'' - a'}$$

$$\therefore \quad \angle (s, r) = \tan.^{-1} \frac{b}{a} = \tan.^{-1} \frac{b'' - b'}{a'' - a'},$$

and $$\angle (s, y_{,}) = \tan.^{-1} \frac{a'' - a'}{b'' - b'}.$$

41. Prob.—*To find the common point of two right lines whose equations are*

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{and} \quad \frac{x}{a'} + \frac{y}{b'} = 1.$$

At the common period the equations are simultaneous

$$\therefore \quad \left.\begin{array}{l} \dfrac{x}{a} + \dfrac{y}{b} = 1 \\ \dfrac{x}{a'} + \dfrac{y}{b'} = 1 \end{array}\right\}$$

$$\therefore \quad \left(\frac{1}{a b'} - \frac{1}{a' b}\right) x = \frac{1}{b'} - \frac{1}{b} = \frac{b - b'}{b b'}$$

$$\therefore \quad x = \frac{a a' (b - b')}{a' b - a b'} = \frac{b - b'}{\dfrac{b}{a} - \dfrac{b'}{a'}}.$$

Similarly $$y = \frac{a - a'}{\dfrac{a}{b} - \dfrac{a'}{b'}}.$$

$\therefore$ the point is

$$\left(\frac{b-b'}{\frac{a}{b}-\frac{a'}{b'}}, \frac{a-a'}{\frac{a}{b}-\frac{a'}{b'}}\right)$$

42. PROB.—*Given the points* (a', b'), (a'', b'') *in one right line, and* $(a_{,}, b_{,})$, $(a_{,,}, b_{,,})$ *in another, to find their common point.*

Their equations are easily found to be

$$\left.\begin{aligned} \frac{x}{\frac{a'b''-a''b'}{b''-b'}} + \frac{y}{\frac{a'b''-a''b'}{a''-a'}} &= 1 \\ \frac{x}{\frac{a_{,}b_{,,}-a_{,,}b_{,}}{b_{,,}-b_{,}}} + \frac{y}{\frac{a_{,}b_{,,}-a_{,,}b_{,}}{a_{,,}-a_{,}}} &= 1 \end{aligned}\right\}$$

and then, as in Art. 41, we get the co-ordinates of the point required.

As a numerical instance, let the points of one right line be (1, 2), (3, 4) and of the other (2, 1), (4, 3). Their common point is required.

Assume their equations to be

$$\left.\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 1 \\ \frac{x}{a'} + \frac{y}{b'} &= 1 \end{aligned}\right\}$$

Then we have

$$\left.\begin{aligned} \frac{1}{a} + \frac{2}{b} &= 1 \\ \frac{3}{a} + \frac{4}{b} &= 1 \end{aligned}\right\} \qquad \left.\begin{aligned} \frac{2}{a'} + \frac{1}{b'} &= 1 \\ \frac{4}{a'} + \frac{3}{b'} &= 1 \end{aligned}\right\}$$

$$\text{or} = \frac{1 + \dfrac{b\,b'}{a\,a'}}{\sqrt{\left\{\left(1 + \dfrac{b^2}{a^2}\right)\ \left(1 + \dfrac{b'^2}{a'^2}\right)\right\}}}$$

THEOREM.—*The angle between two right lines,* $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{a'} + \frac{y}{b'} = 1$ *is that whose tangent is the ratio of the common ordinates with the axes of* y *and* x *respectively of the one* MINUS *the ratio of those respectively of the other, all divided by (unit* PLUS *the product of those ratios.)*

44. PROB.—*Given two points* (a', b') , $(a'' b'')$ *in one right line and two points* $(a_{,} b_{,})$ $(a_{,,}, b_{,,})$ *in another to find the angle between them.*

RULE.—First find the equations to them, which reduce to the form

$$\frac{x}{a} + \frac{y}{b} = 1$$

then by the last theorem the angle required may be found.

EXAMPLE.

Let the given points in one line be (1, 2), (3, 4); and those in the other be (– 2, – 3), (– 5, – 4); then the equations are found to be

$$\frac{x}{-1} + \frac{y}{1} = 1$$

$$\frac{x}{7} + \frac{y}{\left(\frac{-7}{3}\right)} = 1.$$

$$\therefore \quad \angle(s, s') = \tan.^{-1}\frac{\frac{1}{-1} - \left(\frac{-7}{3}\right)}{1 + \frac{1}{-1}\cdot\frac{-7}{3}} = \tan.^{-1}\frac{\frac{7}{3} - 1}{1 + \frac{7}{3}}$$

$$= \tan.^{-1}\frac{4}{10} \quad = \tan.^{-1}\frac{2}{5}.$$

45. Prob.—*To find the conditions that two right lines* $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{a'} + \frac{y}{b'} = 1$ *shall be parallel.*

The angle between them is (43)

$$\tan.^{-1}\frac{\frac{a}{b} - \frac{a'}{b'}}{1 + \frac{a}{b}\cdot\frac{a'}{b'}}.$$

But they are parallel

$$\therefore \quad \tan.^{-1}\frac{\frac{b}{a} - \frac{b'}{a'}}{1 + \frac{b}{a}\cdot\frac{b'}{a'}} = 0,$$

$$\therefore \quad \frac{b}{a} - \frac{b'}{a'} = 0,$$

$$\therefore \quad \frac{b}{a} = \frac{b'}{a'}.$$

46. Theorem.—*When two right lines* $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{a'} + \frac{y}{b'} = 1$ *are parallel, the ratio of the common ordinates with one line = the ratio of them in the other line.*

EXAMPLE.

Are the right lines

$$\left.\begin{array}{l} 2x + 3y = 6 \\ \dfrac{x}{3} + \dfrac{y}{2} = 3 \end{array}\right\}$$

parallel?

Since they become $\dfrac{x}{3} + \dfrac{y}{2} = 1$

$$\text{and} \quad \frac{x}{9} + \frac{y}{6} = 1$$

$$\therefore \quad \frac{b}{a} = \frac{2}{3} \text{ and } \frac{b'}{a'} = \frac{6}{9} = \frac{2}{3}.$$

$\therefore$ they are parallel.

But if any equations to right lines be proposed it is best to find the angle between them at once.

47. PROB.—*To find the conditions that two right lines,* $\dfrac{x}{a} + \dfrac{y}{b} = 1$, $\dfrac{x}{a'} + \dfrac{y}{b'} = 1$ *may be at right angles.*

In this case (43)

$$\tan.^{-1} \frac{\dfrac{b}{a} - \dfrac{b'}{a'}}{1 + \dfrac{b}{a} \cdot \dfrac{b'}{a'}} = \frac{\pi}{2}$$

$$\therefore \quad \frac{\dfrac{b}{a} - \dfrac{b'}{a'}}{1 + \dfrac{b}{a} \cdot \dfrac{b'}{a'}} = \tan. \frac{\pi}{2} = \infty$$

$$1 + \frac{b}{a} \cdot \frac{b'}{a'} = 0.$$

$$\therefore \quad \frac{b'}{a'} = -\frac{1}{\left(\frac{b}{a}\right)}.$$

Hence

48. Theorem.—*When one right line is at right angles to another, the ratio of the common ordinates of one line* = minus *the reciprocal of that of the other line.*

It is often required, at examinations, to solve this problem independently. We shall therefore give a direct investigation.

To find the conditions that the two right lines $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{a'} + \frac{y}{b'} = 1$ *shall be at right angles.* (Fig. 15.)

Let A B, A′ B′ be the lines making a right angle at C; then

$$\angle \text{CED} = \frac{\pi}{2} - \angle \text{CDE} = \angle \text{CDO} - \frac{\pi}{2};$$

$$\therefore \quad \frac{b'}{a'} = \tan.\text{CED} = \tan.\left(\text{CDO} - \frac{\pi}{2}\right) =$$

$$-\frac{\tan.\frac{\pi}{2}}{\tan.\frac{\pi}{2}.\tan.\text{CDO}} = -\frac{1}{\tan.\text{CDO}} = -\frac{1}{\left(\frac{b}{a}\right)}.$$

49. Prob.—*To find the equation of the right line which shall pass through a given point* (a, b), *and make a given angle* (α) *with the given right line*

$$\frac{x}{a'} + \frac{y}{b'} = 1.$$

Assume $$\frac{x}{a''} + \frac{y}{b''} = 1$$

to be the equation required, in which a'', b'' are to be determined. Since the right line is to pass through (a, b) we have

$$\left.\begin{array}{l} \dfrac{a}{a''} + \dfrac{b}{b''} = 1 \\ \text{Also (43) } \tan.\,\alpha = \dfrac{\dfrac{b'}{a'} - \dfrac{b''}{a''}}{1 + \dfrac{b'}{a'} \cdot \dfrac{b''}{a''}} \end{array}\right\}$$

which two equations being thus arranged and solved

$$\left.\begin{array}{r} a \cdot \dfrac{1}{a''} + b \cdot \dfrac{1}{b''} = 1 \\ (a' + b' \tan.\,\alpha)\,\dfrac{1}{a''} + (a' \tan.\,\alpha - b') \cdot \dfrac{1}{b''} = 0 \end{array}\right\}$$

$$\left.\begin{array}{r} \dfrac{a' + b' \tan.\,\alpha}{a''} + \dfrac{a' \tan.\,\alpha - b'}{b''} = 0 \\ \text{and} \quad a \cdot \dfrac{1}{a''} + b \cdot \dfrac{1}{b''} = 1 \end{array}\right\},$$

$$\therefore \; (a\,a' \tan.\,\alpha - a\,b' - a'\,b - b\,b' \tan.\,\alpha)\,\frac{1}{a''}$$
$$= a' \tan.\,\alpha - b',$$

$$\therefore \; \frac{1}{a''} = \frac{a' \tan.\,\alpha - b'}{(a\,a' - b\,b') \tan.\,\alpha - (a\,b' + a'\,b)}$$

$$= \frac{1}{a + b \cdot \dfrac{a' + b' \tan.\,\alpha}{b' - a' \tan.\,\alpha}}$$

Similarly $\quad \dfrac{1}{b''} = \dfrac{1}{b + a \, . \, \dfrac{b' + a' \tan. \alpha}{a' - b' \tan. \alpha}}$

which give the required equation.

Ex. 1. *To find the equation of a right line passing through the point* (2, 3) *and making an angle of* 45° *with the right line* $\dfrac{x}{4} + \dfrac{y}{5} = 1$.

Assume the equation to be

$$\frac{x}{a} + \frac{y}{b} = 1;$$

then
$$\frac{2}{a} + \frac{3}{b} = 1,$$

and
$$\frac{\frac{5}{4} - \frac{b}{a}}{1 + \frac{5}{4} \cdot \frac{b}{a}} = \tan. 45^\circ = 1$$

$$\therefore \quad \frac{9}{4} \cdot \frac{b}{a} = \frac{5}{4} - 1 = \frac{1}{4},$$

$$\therefore \quad a = 9b$$

Hence
$$\frac{1}{a} = \frac{1}{29} \text{ and } \frac{1}{b} = \frac{9}{29};$$

$$\therefore \quad \frac{x}{29} + \frac{9y}{29} = 1$$

is the equation required.

Ex. 2. – *To find the equation of the right line passing through the point* (a, b) *and making right angles with the right line* $\dfrac{x}{a'} + \dfrac{y}{b'} = 1$.

Assuming the equation to be

$$\frac{x}{a''} + \frac{y}{b''} = 1,$$

We have $$\frac{a}{a''} + \frac{b}{b''} = 1,$$

and $$\frac{\frac{b'}{a'} - \frac{b''}{a''}}{1 + \frac{b'}{a''} \cdot \frac{b''}{a}} = \tan. 90^\circ = \infty;$$

$$\therefore \quad \frac{b''}{a''} = -\frac{a'}{b'}$$

$$\therefore \quad \frac{1}{a''} = -\frac{a'}{b'} \cdot \frac{1}{b''}.$$

Hence by substitution

$$\left(b - \frac{a a'}{b'}\right)\frac{1}{b''} = 1;$$

$$\therefore \quad \frac{1}{b''} = \frac{1}{b - \frac{a a'}{b'}}$$

$$\therefore \quad \frac{1}{a''} = \frac{a'}{a a' - b b'} = \frac{1}{a - \frac{b b'}{a'}}.$$

$\therefore$ the equation required is

$$\frac{x}{a - b \cdot \frac{b'}{a'}} + \frac{y}{b - a \cdot \frac{a'}{b'}} = 1.$$

50. Theorem.—*If a right line pass through a given point at right angles to a given right line, the common ordinate of* x *of the former right line is the common ordinate of* x *in the latter,* minus *the product of the common ordinate of* y *in the given right line, and of the ratio*

of the co-ordinates y *and* x *respectively of the given point.*

Also, the common ordinate of y *of the former right line is that of* y *in the given right line,* minus *the product of the common ordinate of* x *in the given right line, and the ratio of the co-ordinates* x, y *respectively of the given points.*

Ex. 3.—*To find the equation of the right line which passes through* (a, b) *and is parallel to a given right line* $\frac{x}{a'} + \frac{y}{b'} = 1$.

It is $$\frac{x}{a - b \cdot \frac{a'}{b'}} + \frac{y}{b - a \cdot \frac{b'}{a'}} = 1.$$

51. Prob.—*To find the length of the perpendicular drawn from a given point* (a, b) *upon a given right line* $\frac{x}{a'} + \frac{y}{b'} = 1$.

The distance from the given point (a, b) to any point (x, y) in the given right line, is

$$s = \sqrt{\{(x - a)^2 + (y - b)^2\}}$$

But assuming $$\frac{x}{a''} + \frac{y}{b''} = 1$$

as the equation of s, we get, from the right lines being at right angles, (43)

$$\frac{\frac{b'}{a'} - \frac{b''}{a''}}{1 + \frac{b'}{a'} \cdot \frac{b''}{a''}} = \tan\frac{\pi}{2} = \infty\ ;$$

$$\therefore \quad \frac{b''}{a''} = -\frac{a'}{b'} \quad \ldots \ldots \quad (1).$$

And from the perpendicular also passing through the point (a, b) we have

$$\frac{a}{a''} + \frac{b}{b''} = 1 \quad \ldots \ldots \ldots \quad (2).$$

whence
$$\frac{1}{a''} = \frac{a'}{a\,a' - b\,b'},$$

$$\frac{1}{b''} = -\frac{b'}{a\,a' - b\,b'}.$$

And the assumed equation becomes, at the common point of the two right lines,

$$\left.\begin{array}{ll} & \dfrac{a'}{a\,a' - b\,b'}\,x - \dfrac{b'}{a\,a' - b\,b'}\,y = 1 \\ \text{also} & \dfrac{x}{a'} + \dfrac{y}{b'} = 1 \end{array}\right\},$$

$$\left.\begin{array}{ll} \text{whence} & (x-a)^2 = \dfrac{b'^2\,(a\,b' + a'b' - a'\,b')^2}{(a'^2 + b'^2)^2}, \\ \text{and} & (y-b)^2 = \dfrac{a'^2\,(a'\,b + a\,b' - a'\,b')^2}{(a'^2 + b'^2)^2}. \end{array}\right\}$$

$\therefore$ the distance required is,

$$s = \sqrt{\frac{(a'^2 + b'^2)\,(a\,b' + a'b - a'\,b')^2}{(a'^2 + b'^2)^2}}$$

$$\text{or,} \quad \frac{a\,b' + a'\,b - a'\,b'}{\sqrt{(a'^2 + b'^2)}}$$

$$\text{or,} \quad \frac{\dfrac{a}{a'} + \dfrac{b}{b'} - 1}{\sqrt{\left(\dfrac{1}{a'^2} + \dfrac{1}{b'^2}\right)}}.$$

This process is not difficult, nor is it inelegant or indirect, but the contrary. There is another method, however, which has the advantage in point of brevity (Fig. 16):—

Let A B be the given straight line cutting the axis of x in A, and C the given point. Draw C M $\perp$ O X cutting A B in Q, C N $\perp$ A B. Then,

$$\begin{aligned} \text{C N} &= \text{C Q sin. N Q C} \\ &= (\text{C M} - \text{Q M}) \text{ cos. O A B} \\ &= (b - \text{A M tan. O A B}) \text{ cos. O A B} \\ &= \{b - (\text{O A} - \text{O M}) \text{ tan. O A B}\} \text{ cos. O A B} \\ &= \left\{b - (a' - a)\frac{b'}{a'}\right\} \frac{1}{\sqrt{\left(1 + \frac{b'^2}{a'^2}\right)}} \\ &= \frac{a b' + a' b - a' b'}{\sqrt{(a'^2 + b'^2)}} = \frac{\frac{a}{a'} + \frac{b}{b'} - 1}{\sqrt{\left(\frac{1}{a'^2} + \frac{1}{b'^2}\right)}}, \end{aligned}$$

the value required.

52. THEOREM.—*The length of the perpendicular from a given point upon a given right line is the sum of the ratios of each of the co-ordinates of the given points to each of the corresponding common ordinates of the right line* minus *unit, all divided by the square root of the sum of the squares of the reciprocals of the common ordinates of the line.*

These are the most useful and obvious problems de-

ducible from the theory of the right line; but there are innumerable others, which are in general mere puzzles, or matters of curiosity. We will give a specimen:—

Given (a, b), (a', b'), (a'', b''), *the angular points of a* Δ *to find its area.*

Given the equations of the sides of a Δ, viz.,

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{a'} + \frac{y}{b'} = 1, \quad \frac{x}{a''} + \frac{y}{b''} = 1,$$

to find its area.

SECTION III.

THEORY OF THE CIRCLE IN A CO-ORDINATE PLANE.

53. DEF.—*A circle is a line every point of which is in the same plane, and equidistant from a given point in that plane; which given point is called the* CENTRE. *The equal distances are also termed* RADIUS.

Euclid's definition of a circle is objectionable, for several reasons; but the principal defect consists in his having called the whole surface, or area of the circle, the circle itself. It is merely the line, that he calls the circumference, which is the circle.

54. PROP.—*Given the co-ordinates of the centre* (a, b) *of a circle, and the radius* R, *to find the rectangular equation of it* (Fig. 17):—

Let C be the centre and P any point (x, y) of the

circle ; then P M, C D being parallel to O Y and C M to O X, we have

$$C P^2 = C m^2 + P m^2,$$

or, $$R^2 = (x - a)^2 + (y - b)^2,$$

$$\therefore \quad \frac{(x-a)^2}{R^2} + \frac{(y-b)^2}{R^2} = 1 \ldots\ldots (1)$$

the equation required.

55. THEOREM.—*The equation of a circle is the sum of the squares of the ratios of the differences of the corresponding co-ordinates of the centre and any point of the circle, to the radius, all equal to unit.*

EXAMPLE.

To find the equation of the circle whose centre is (2, 3) *and radius* = 5.

By 55.

$$\frac{(x-2)^2}{25} + \frac{(y-3)^2}{25} = 1$$

which is reducible to

$$x^2 + y^2 - 4x - 6y = 12.$$

Conversely, by completing the squares of the parts $x^2 - 4x$, $y^2 - 6y$ we shall reduce this latter equation to that form in which we perceive the values of the co-ordinates of the centre and of the radius. Thus we get

$$x^2 - 4x + 4 + y^2 - 6x + 9 = 12 + 4 + 9 = 25;$$

$$\therefore \quad \frac{(x-2)^2}{25} + \frac{(y-3)^2}{25} = 1.$$

56. PROP.—*Given the polar co-ordinates* (R′, α) *of the centre of a circle and its radius* R, *to find its polar equation.*

Let $OP = r, OC = R'$,

$XOP = \theta, CP = R$, and $\angle COX = \alpha$.

Then $\because$

$$\cos. COP = \frac{OP^2 + CO^2 - CP^2}{2 OP . CO}$$

or, $$\cos. (\theta - \alpha) = \frac{r^2 + R'^2 - R^2}{2 R' . r}$$

$$\therefore \quad r^2 - 2 R' r \cos. (\theta - \alpha) = R^2 - R'^2 \quad \ldots . (2)$$

This equation may also be obtained from the substitution of

$$\left.\begin{array}{l} x = r \cos. \theta, \\ a = R' \cos. \alpha \end{array}\right\} \left.\begin{array}{l} y = r \sin. \theta, \\ b = R' \sin. \alpha, \end{array}\right\}$$

in the equation

$$(x - a)^2 + (y - b)^2 = R^2.$$

For $$x^2 + y^2 + a^2 + b^2 - 2 a x - 2 b y = R^2,$$

$$\therefore r^2 (\cos.^2 \theta + \sin.^2 \theta) + R'^2 (\cos.^2 \alpha + \sin.^2 \alpha)$$
$$- 2 R' r (\cos. \theta \cos. \alpha, + \sin. \theta . \sin. \alpha) = R^2,$$

$$\therefore \quad r^2 - 2 R' r \cos. (\theta - \alpha) = R^2 - R'^2,$$

&c. &c.

Such are the general equations of the circle.

Particular Equations of the Circle.

1. If the origin of co-ordinates be at the centre; then $a = 0, b = 0$, and the equation

$$\frac{(x - a)^2}{R^2} + \frac{(y - b)^2}{R^2} = 1$$

becomes $$\frac{x^2 + y^2}{R^2} = 1 \quad \ldots . . (3).$$

Also, in this case, $R' = 0$ and the polar equation becomes $$r = R \ldots \ldots . . (4).$$

2. If the origin be any point of the circle, then

$$a^2 + b^2 = R^2$$

and $$\frac{(x - a)^2 + (y - b)^2}{R^2} = 1$$

becomes

$$x^2 + y^2 - 2\,a\,x - 2\,b\,y = 0 \;\ldots\ldots\; (5)$$

Also, in this case, $R' = R$ and the polar equation becomes

$$r - 2\,R \cos.\,(\theta - \alpha) = 0 \;\ldots\ldots\; (6)$$

3. If the origin be any point of the circle, and the axis of x pass through the centre;

then $$a = R,\ b = 0,$$

and $$\frac{(x - a)^2 + (y - b)^2}{R^2} = 1$$

becomes $$x^2 + y^2 - 2\,R\,x = 0 \;\ldots\; (7).$$

Also, α is 0, and $\therefore$ the polar equation is

$$r - 2\,R \cos.\,\theta = 0 \;\ldots\ldots\; (8).$$

RECAPITULATION OF THE EQUATIONS OF A CIRCLE.

57.—1. *If the centre be the point* (a, b) *or* (R', α), *then the rectangular and polar equations are respectively*

$$\frac{(x - a)^2}{R^2} + \frac{(y - b)^2}{R^2} = 1$$

and $$\frac{r^2}{R^2 - R'^2} - \frac{2\,R' r \cos.\,(\theta - \alpha)}{R^2 - R'^2} = 1.$$

2. *If the centre be any point* $\{a, \sqrt{(R_2 - a^2)}\}$, *that is, if the origin be any point of the circle, then they are*

$$\left.\begin{array}{ll} & x^2 + y^2 - 2\,a\,x - 2\sqrt{(R^2 - a^2)}\,.\,y = 0 \\ \text{and} & r - 2\,R \cos.\,(\theta - \alpha) = 0. \end{array}\right\}$$

3. *If the centre be the point* (R, 0) *or* (R, 0 polar); *that is, if the origin be any point of the circle, and the axis of* x *pass through the centre, then they are*

$$\left.\begin{aligned} x^2 + y^2 - 2\,R\,x &= 0, \\ \text{and} \qquad r - 2\,R\cos.\,\theta &= 0. \end{aligned}\right\}$$

4. *If the origin be at the centre; they are*

$$\frac{x^2}{R^2} + \frac{y^2}{R^2} = 1,$$

$$\text{and} \qquad r = R.$$

Since the equation

$$\frac{(x-a)^2}{R^2} + \frac{(y-b)^2}{R^2} = 1$$

may be reduced to

$$x^2 + y^2 - 2\,a\,x - 2\,b\,y + a^2 + b^2 - R^2 = 0,$$

it is generally of the form

$$x^2 + y^2 + A\,x + B\,y + C^2 = 0.$$

Hence is suggested the next proposition.

58. PROP.—*Every equation which can be reduced to the form*

$$x^2 + y^2 + A\,x + B\,y + C^2 = 0,$$

is that of a circle, when

$$\frac{A^2}{4} + \frac{B^2}{4} \text{ is } > C^2.$$

For completing the squares of $x^2 + A\,x$, $y^2 + B\,y$ by the addition and subtraction of

$$\frac{A^2}{4} \text{ and } \frac{B^2}{4}$$

we get

$$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 + C^2 - \frac{A^2}{4} - \frac{B^2}{4} = 0;$$

$$\therefore \quad \left(x+\frac{A}{2}\right)^2 + \left(y+\frac{B}{2}\right)^2 = \frac{A^2}{4} + \frac{B^2}{4} - C^2.$$

But the equation of the circle, whose radius is R and centre (a, b), is

$$(x-a)^2 + (y-b)^2 = R^2;$$

$\therefore$ the given equation is that of a circle whose radius is

$$\sqrt{\left(\frac{A^2}{4} + \frac{B^2}{4} - C^2\right)}$$

and whose centre is the point

$$\left(-\frac{A}{2}, -\frac{B}{2}\right).$$

Examples.

Ex. 1. *To find the centre and radius of the circle whose equation is*

$$x^2 + y^2 - 2x + 4y - 49 = 0.$$

Completing the squares of $x^2 - 2x$, $y^2 + 4y$, we get

$$(x-1)^2 + (y+2)^2 = 54;$$

$\therefore$ the radius of the circle is $\sqrt{54}$, and its centre is the point $(1, -2)$.

Ex. 2. *To find the centre and radius of the circle whose equation is*

$$x^2 + y^2 + 2x - 4y + 49 = 0.$$

Here we get

$$(x+1)^2 + (y-2)^2 = -44;$$

which is an absurd result, provided x and y are real quantities. Hence this equation cannot belong to a circle. We shall show hereafter what is the representative of equations of this kind.

Ex. 3. *To find the centre and radius of the circle whose equation is*

$$3x^2 + 3y^2 + 9x + 6y + 2 = 0.$$

First dividing by 3,

$$x^2 + y^2 + 3x + 2y + \frac{2}{3} = 0,$$

and completing the squares

$$\left(x + \frac{3}{2}\right)^2 + (y + 1)^2 = \frac{9}{4} + 1 - \frac{2}{3} = \frac{31}{12};$$

$\therefore$ the centre is

$$\left(-\frac{3}{2}, -1\right) \text{ and rad.} = \frac{1}{2}\sqrt{\frac{31}{3}}.$$

Ex. 4. *The centre and radius of the circle whose equation is*

$$x^2 + y^2 - 2x + 4y - 7 = 0,$$

are respectively $(1, -2)$ and $2\sqrt{3}$.

Ex. 5 *The centre and radius of the circle whose equation is*

$$x^2 + y^2 + 4y + 3 = 0,$$

are respectively $(0, -2)$ and 1.

Ex. 6. *The centre and radius of the circle whose equation is*

$$x^2 + y^2 - 8x + 9 = 0,$$

are respectively $(4, 0)$ and $\sqrt{7}$.

Ex. 7. *The centre and radius of the circle whose equation is*

$$x^2 + y^2 - 16 = 0,$$

are respectively $(0, 0)$ and 4.

Ex. 8. *The centre and radius of the circle whose equation is*

$$x^2 + y^2 + 3x = 0,$$

are respectively $\left(-\frac{3}{2}, 0\right)$ and $\frac{3}{2}$.

Ex. 9. *The centre and radius of that whose equation is*

$$x^2 + y^2 - 4y = 0,$$

are respectively (0, 2) and 2.

Ex. 10. *The centre and radius of that whose equation is*

$$A x^2 + A y^2 + B x + C y = 0,$$

are respectively

$$\left(-\frac{B}{2A}, -\frac{C}{2A}\right) \text{ and } \frac{\sqrt{(B^2 + C^2)}}{2A}.$$

To the previous proposition we may add this one.

Every equation which can be reduced to the form

$$r^2 + A r \cos.(\theta - \alpha) + B = 0,$$

in which r and θ are the only variables, is the polar equation of a circle; when B is negative, or positive, and less than $\frac{A^2}{4}$.

For the polar equation of a circle whose centre is (R', α) and rad. $= R$ is (Art. 56.)

$$r^2 + 2R' r.\cos.(\theta - \alpha) + R'^2 - R^2 = 0;$$

$$\therefore \quad -2R' = A, \text{ and } R'^2 - R^2 = B;$$

$$\therefore \quad R' = -\frac{A}{2}, \text{ and } R^2 = R'^2 - B = \frac{A^2}{4} - B;$$

$\therefore$ the equation is that of a circle whose centre is $\left(\frac{A}{2}, \alpha\right)$ and radius $= \sqrt{\left(\frac{A^2}{4} - B\right)}$.

If B be positive and $= \frac{A^2}{4}$ there is no radius.

If B be positive and $> \frac{A^2}{4}$ there is no radius.

Examples.

Ex. 1. *Find the centre and radius of the circle whose equation is*

$$r^2 + 3\,r \cos.(\theta - 30^\circ) - 9 = 0.$$

Since the general form of the equation of a circle is

$$r^2 - 2\,R'\,r \cos.(\theta - \alpha) + R'^2 - R^2 = 0$$

$$\therefore\ R' = \frac{3}{2} \text{ and } R^2 = \frac{9}{4} + 9$$

$\therefore$ the centre is $\left(\frac{3}{2}, 30^\circ\right)$ and radius $= \frac{3}{2}\sqrt{5}$.

Ex. 2. *Find the centre and radius of that whose equation is*

$$4\,r^2 - 9\,r \cos.50^\circ \sin.\theta - 9\,r \sin.50^\circ \cos.\theta + 4 = 0.$$

First reducing the equation to the requisite form, we get

$$r^2 - \frac{9}{4}\,r\,(\cos.\theta.\cos.40^\circ + \sin.\theta \sin.40^\circ) + 1 = 0.$$

$$\text{or } r^2 - \frac{9}{4}\,r \cos.(\theta - 40^\circ) + 1 = 0.$$

Hence as above

$$R' = \frac{9}{8}\,,\text{ and } R^2 = \frac{81}{64} - 1 = \frac{17}{64}$$

$\therefore$ the centre is $\left(\frac{9}{8}, 40°\right)$ and radius $= \frac{1}{8}\sqrt{17}$.

Ex. 3. *Find the centre and radius of the circle whose equation is*

$$r + 3\cos.(\theta - 45°) = 0.$$

Comparing it with

$$r^2 - 2R'r\cos.(\theta - \alpha) + R'^2 - R^2 = 0,$$

$$R' = \frac{3}{2}, \ R'^2 - R^2 = 0$$

$$\text{or} \quad R = R' = \frac{3}{2};$$

$\therefore$ the circle is that whose centre is

$$\left(\frac{3}{2}, 45°\right) \text{ and radius } = \frac{3}{2}.$$

Ex. 4. *To construct the circle whose equation is*

$$r - 3\cos.\theta = 0.$$

Its centre is $\left(\frac{3}{2}, 0\right)$ and radius $= \frac{3}{2}$.

Ex. 5. *To construct the circle whose equation is*

$$r = 5.$$

Its centre is the origin of co-ordinates and radius $= 5$.

Ex. 6. *To construct the equation,*

$$2r^2 - 4r\cos.\theta + 11 = 0.$$

First preparing it

$$r^2 - 2r\cos.\theta + \frac{11}{2} = 0,$$

then

$$R' = 1, \text{ and } R^2 = R'^2 - \frac{11}{2} = 1 - \frac{11}{2} = -\frac{9}{2}.$$

$\therefore$ R is imaginary. That is, the values of r and θ in

this equation cannot be both real, and consequently the equation cannot belong to a circle.

Problems on Points, Right Lines, and Circles.

59. Prob.—*To find the common points of a circle* $(x - a)^2 + (y - b)^2 = R^2$ *with the co-ordinate axes.*

$$\text{Let} \quad x = 0,$$

$$\text{then} \quad (y - b)^2 = R^2 - a^2,$$

$$\text{and} \quad y = b \pm \sqrt{(R^2 - a^2)}.$$

$\therefore$ the circle meets the axis of y in the points

$$\{o, b + \sqrt{(R^2 - a^2)}\}, \{o, b - \sqrt{(R^2 - a^2)}\}.$$

Let $y = 0$; then in like manner

$$x = a \pm \sqrt{(R^2 - b^2)},$$

and the common points of the axis of x and the circle are

$$\{a + \sqrt{(R^2 - b^2)}, o\}, \{a - \sqrt{(R^2 - b^2)}, o\}.$$

If a be $>$ R, $\sqrt{(R^2 - a^2)}$ is imaginary, and $\therefore$ the circle does not meet the axis of y; for the same reason if b be $>$ R it does not meet the axis of x; that is, in such cases the circle and the axes have no common points.

If $a = R$, then the circle and axis of y have one common point, and since $y = b$, if this point be joined with the centre, the joining radius will be $\perp$ to the axis, and consequently the axis of y in this case touches the circle.

Similarly when $b = R$, the axis of x touches the circle.

60. Prob.—*To find the common points of a circle* $r^2 - 2R'r\cos.(\theta - \alpha) = R^2 - R'^2$ *with the axis.*

$$\text{Let} \quad \theta = 0;$$

$$\text{then} \quad r^2 - 2\,R' r \cos.\,\alpha = R^2 - R'^2;$$

$$\therefore \; r = R' \pm \sqrt{(R^2 - R'^2 + R'^2 \cos.^2 \alpha)}$$
$$= R' \pm \sqrt{(R^2 - R'^2 \sin.^2 \alpha)}.$$

$\therefore$ the common points with the axis of x, determined by polar co-ordinates, are

$$\{R' + \sqrt{(R^2 - R'^2 \sin.^2 \alpha)}\;, o\}, \{R' - \sqrt{(R^2 - R'^2 \sin.^2 \alpha)}, o\}$$

$$\text{Let} \quad \theta = \frac{\pi}{2};$$

$$\text{then} \quad r^2 - 2\,R' r \sin.\,\alpha = R^2 - R'^2;$$

$$\therefore \; r = R' \pm \sqrt{(R^2 - R'^2 \cos.^2 \alpha)}$$

and the common points with the axis of y are

$$\left\{R' + \sqrt{(R^2 - R'^2 \cos.^2 \alpha)}\;, \frac{\pi}{2}\right\},$$

$$\left\{R' - \sqrt{(R^2 - R'^2 \cos.^2 \alpha)}\;, \frac{\pi}{2}\right\}.$$

If R be $>$ R$'$ sin. α, the circle does not meet the axis of x.

If R be $>$ R$'$ cos. α, the circle does not meet the axis of y.

If R $=$ R$'$ sin. α, the axis of x touches the circle.

If R $=$ R$'$ cos. α, ... y . . . .

61. Prob.—*Given the co-ordinates of the centre of a circle and its radius to find the common points of the circle and axes.*

First find the equation to the circle and then proceed as in Art. 59 or 60.

62. PROB.—*To find the common points of a right line* $\frac{x}{a} + \frac{y}{b} = 1$, *and a circle* $(x - a')^2 + (y - b')^2 = R^2$.

At the common points the equations are simultaneous.

Solving these equations, we first get

$$y = b - \frac{b}{a}x;$$

$$\therefore \quad (x - a')^2 + \left(b - b' - \frac{b}{a}x\right)^2 = R^2$$

$$\therefore \quad x^2 - 2a'x + a'^2 + (b - b')^2 - 2\frac{b}{a}(b - b')x + \frac{b^2}{a^2}x^2 = R^2,$$

$$\therefore \quad \frac{a^2 + b^2}{a^2}x^2 - 2\frac{aa' - bb' + b^2}{a}x = R^2 - a'^2 - (b - b')^2$$

$$\therefore \quad x^2 - 2a \, . \, \frac{aa' - bb' + b^2}{a^2 + b^2}x = a^2\frac{R^2 - a'^2 - (b - b')^2}{a^2 + b^2};$$

which equation being solved gives

$$x = a \, . \, \frac{aa' - bb' + b^2}{a^2 + b^2} \pm \frac{a}{a^2 + b^2} \times \sqrt{[(aa' - bb' + b^2)^2 + \{(R^2 - a'^2) - (b - b')^2\}(a^2 + b^2)]}$$

$$= \frac{a}{a^2 + b^2}[(aa' - bb' + b^2) \pm \sqrt{\{(a^2 + b^2)R^2 + 2ab(ab' + a'b) - 2aa'bb' - a^2(b^2 + b'^2) - a'^2b^2\}}].$$

Hence may be found a similar pair of simultaneous values for y and then two pairs of values, except when they are imaginary, in which case the right line and

circle have no common points, will be the co-ordinates of two common points of the line and circle. They touch when the surd part of the expression vanishes.

EXAMPLE.

To find the common points of the right line $\frac{x}{2} + \frac{y}{3} = 1$ *and the circle* $(x - 1)^2 + (y - 4)^2 = 9$.

Here $$y = 3 - \frac{3}{2}x;$$

$\therefore$ at the common points

$$(x - 1)^2 + \left(\frac{3x}{2} + 1\right)^2 = 9;$$

$$\therefore \quad x^2 - 2x + 1 + \frac{9x^2}{4} + 3x + 1 = 9,$$

$$\therefore \quad \frac{13}{4}x^2 + x = 7,$$

$$\therefore \quad x^2 + \frac{4}{13}x = \frac{28}{13},$$

$$\therefore \quad x = -\frac{2}{13} \pm \sqrt{\left(\frac{4}{13^2} + \frac{28}{13}\right)} = \frac{-2 \pm 4\sqrt{23}}{13},$$

Hence $$y = 3 - \frac{3}{2}x = 3 + \frac{2 \mp 4\sqrt{23}}{13}$$

$$= \frac{41 \mp 4\sqrt{23}}{13}$$

$\therefore$ the common points are

$$\left(\frac{-2 + 4\sqrt{23}}{13}, \frac{41 - 4\sqrt{23}}{13}\right)$$
$$\left(\frac{-2 - 4\sqrt{23}}{13}, \frac{41 + 4\sqrt{23}}{13}\right).$$

63. PROB.—*To find the common points of two circles whose equations are*

$$(x - a)^2 + (y - b)^2 = R^2,$$
$$(x - a')^2 + (y - b')^2 = R'^2.$$

At the common points the equations are simultaneous. These equations become

$$\left.\begin{array}{l} x^2 + y^2 - 2ax - 2by = R^2 - a^2 - b^2 \\ x^2 + y^2 - 2a'x - 2b'y = R'^2 - a'^2 - b'^2 \end{array}\right\} \ldots (a)$$

$\therefore$ by subtraction

$$2(a - a')x + 2(b - b')y = R'^2 - R^2 + a^2 - a'^2 + b^2 - b'^2,$$

$$\therefore \quad y = \frac{R'^2 - R^2 + a^2 - a'^2 + b^2 - b'^2}{2(b - b')} - \frac{a - a'}{b - b'}x.$$

$\therefore$ by substitution in the first of equations (a) we shall get a quadratic of the form

$$x^2 + Ax + B = 0,$$

which being solved will either give two unequal real values, two equal real values, or two imaginary values of x, and these being substituted in either of equations (a) will give an equation of the same form as that for x, and the values of y will be similar to those of x. Hence, either two common points of the circles will be obtained, or one point only at which they touch; or the result will show that they have no common point according to these respective cases.

EXAMPLE.

To find the common points of the circles $x^2 + y^2 = 1$, and $(x - 1)^2 + (y - 2)^2 = 4$.

$$\left.\begin{array}{lr} \text{First} & x^2 + y^2 - 2x - 4y = 4 - 5 = -1 \\ \text{and} & x^2 + y^2 \ldots\ldots\ldots\ldots = 1 \end{array}\right\}$$

$$\therefore\ 2x + 4y = 2,$$

and $$x + 2y = 1;$$

$\therefore\ x = 1 - 2y$, and substituting in $x^2 + y^2 = 1$,

$$1 - 4y + 4y^2 + y^2 = 1,$$

$$\therefore\ y^2 - \frac{4}{5}y = 0, \qquad \therefore\ y = 0 \text{ and } \frac{4}{5},$$

$$\therefore\ x = 1 - 2y = 1 \text{ and } 1 - \frac{8}{5}$$

$$= 1 \text{ and } \frac{-3}{5},$$

$\therefore$ the common points are $(1, 0)$, $\left(\frac{-3}{5}, \frac{4}{5}\right)$.

If the pairs of values had been equal, the common point would have been one of contact. If they had been imaginary, it would have proved the circles to have neither cut nor touched.

64. Prob.—*To find the equation of the right line joining the centres of two circles whose equations are* $(x - a)^2 + (y - b)^2 = R^2$, $(x - a')^2 + (y - b')^2 = R'^2$.

The co-ordinates of their centres being $(a, b,)$ (a', b') if we assume the equation required to be

$$\frac{x}{a''} + \frac{y}{b''} = 1,$$

we have at the centres

$$\left.\begin{array}{l} x = a \\ y = b \end{array}\right\} \quad \left.\begin{array}{l} x = a' \\ y = b' \end{array}\right\} \quad \text{and}$$

$$\therefore \quad \left.\begin{array}{l} \dfrac{a}{a''} + \dfrac{b}{b''} = 1 \\[2ex] \dfrac{a'}{a''} + \dfrac{b'}{b''} = 1 \end{array}\right\} .$$

Whence $\dfrac{1}{a''}$ and $\dfrac{1}{b''}$ being found as in simple equations to be

$$\frac{b' - b}{ab' - a'b}, \; \frac{a' - a}{ab' - ab'},$$

the equation required is

$$\frac{b' - b}{ab' - a'b}\,x - \frac{a' - a}{ab' - a'b}\,y = 1.$$

Prob.—*To find the equation of the right line joining the common points of two circles whose equations are* $(x - a)^2 + (y - b)^2 = R^2$; $(x - a')^2 + (y - b')^2 = R'^2$.

Find the co-ordinates of those points and thence the equation of the right line that passes through them.

65. **Def.**—*The* SECANT *at any point of a curve is that right line which passes through that and any other points of the curve.*

66. **Def.**—*The* TANGENT *at any point of a curve is that right line which the secant becomes when that other point is supposed to coincide with the given point.*

67. **Def.**—*The* NORMAL *at any point of a curve is the right line drawn from that point at right angles to the tangent at that point.*

68. **Prob.**—*To find the equation of a tangent of a circle* $x^2 + y^2 = R^2$, *at the point* (a, b) *of it.*

Let a secant $\frac{x}{a''} + \frac{y}{b''} = 1$, pass through the points (a, b), (a', b'), it will become the tangent required when a'', b'' are found in terms a, b, a', b' and the point (a', b') is supposed to coincide with $(a\,b)$.

For this purpose we have

$$\left.\begin{array}{l} \frac{a'}{a''} + \frac{b'}{b''} \\ \frac{a}{a''} + \frac{b}{b''} \end{array}\right\} \quad \left.\begin{array}{l} a'^2 + b'^2 = R^2 \\ a^2 + b^2 = R^2 \end{array}\right\}.$$

From the first pair we have

$$\frac{1}{a''} = \frac{b' - b}{a\,b' - a'\,b};$$

$$\therefore\ a'' = \frac{a\,b' - a'\,b}{b' - b} = a - b \,.\, \frac{a' - a}{b' - b} \text{ by division.}$$

But from the second pair

$$a'^2 - a^2 + b'^2 - b^2 = 0,$$

$$\therefore\ \frac{a' - a}{b' - b} = -\frac{b' + b}{a' + a};$$

$$\therefore\ a'' = a + b \,.\, \frac{b' + b}{a' + a} = \frac{a^2 + b^2 + a\,a' + b\,b'}{a + a'}.$$

$$\text{Similarly} \qquad b'' = \frac{a^2 + b^2 + a\,a' + bb'}{b + b'},$$

$\therefore$ the equation of the secant is

$$\frac{x}{\left(\frac{a^2 + b^2 + a\,a' + b\,b'}{a + a'}\right)} + \frac{y}{\left(\frac{a^2 + b^2 + a\,a' + b\,b'}{b + b'}\right)} = 1.$$

Let the point (a', b') coincide with (a, b); then the equation of the tangent is

$$\frac{x}{\left(\frac{a^2 + b^2}{a}\right)} + \frac{y}{\left(\frac{a^2 + b^2}{b}\right)} = 1;$$

$$\text{or} \quad \frac{x}{\left(\frac{R^2}{a}\right)} + \frac{y}{\left(\frac{R^2}{b}\right)} = 1.$$

69. Prob.—*To find the equation of the Normal of a circle*

$$x^2 + y^2 = R^2,$$

at any point (a, b) of it.

Let $\frac{x}{a'} + \frac{y}{b'} = 1$ be the equation of the tangent,

$\frac{x}{a''} + \frac{y}{b''} = 1$ that of the normal.

Then $\because$ the normal passes through (a, b) and is $\perp$ to the tangent,

$$\therefore \quad a'' = a - b \cdot \frac{b'}{a'}, \; b'' = b - a \cdot \frac{a'}{b'}.$$

$$\text{But} \quad a' = \frac{R^2}{a}, \; b' = \frac{R^2}{a};$$

$$\therefore \quad a'' = a - b \cdot \frac{a}{b} = 0, \text{ and } b'' = b - a \cdot \frac{b}{a} = 0;$$

$\therefore$ the normal passes through the origin of co-ordinates, and the equation of the normal is

$$x + \frac{a''}{b''} \cdot y = 0,$$

$\frac{a''}{b''}$ being a vanishing fraction. But

$$\frac{a''}{b''} = \frac{a - b \cdot \dfrac{b'}{a'}}{b - a \cdot \dfrac{a'}{b'}} = - \frac{a\,a' - b\,b'}{a\,a' - b\,b'} \cdot \frac{b'}{a'}$$

$$= - \frac{b'}{a'} = - \frac{a}{b};$$

$\therefore$ the equation required of the normal is

$$\frac{x}{a} - \frac{y}{b} = 0.$$

Otherwise. (Fig. 18.)

Let $T\,T'$ be the tangent of the circle at its point P, and N N′ the corresponding normal passing through its centre O. Then P′ being any point (x, y) of the normal, by similar $\triangle^s$ we have

$$P'M' : O\,M' :: P\,M : O\,M,$$

Or $\qquad y : x \quad :: b \quad : a,$

$$\therefore \qquad \frac{x}{a} = \frac{y}{b} \text{ as before.}$$

Observe, also, that since O A, O B are the common ordinates of the tangent, the equation of this tangent is

$$\frac{x}{O\,A} + \frac{y}{O\,B} = 1;$$

But $\qquad O\,A : O\,P :: O\,P : O\,M;$

$$\therefore \qquad O\,A = \frac{O\,P^2}{O\,M} = \frac{R^2}{a};$$

Similarly. $\qquad O\,B = \frac{O\,P^2}{P\,M} = \frac{R^2}{b};$

and the equation is

$$\frac{x}{\left(\frac{R^2}{a}\right)} + \frac{y}{\left(\frac{R^2}{b}\right)} = 1, \text{ as before.}$$

70. **Prob.**—*To find the equation of the Tangent at any point (a, b) of the circle,*

$$(x - A)^2 + (y - B)^2 = R^2.$$

Let the secant $\frac{x}{a''} + \frac{y}{b''} = 1$ pass through the points (a, b), (a', b'); then

$$\left.\begin{aligned}\frac{a}{a''} + \frac{b}{b''} &= 1\\ \frac{a'}{a''} + \frac{b'}{b''} &= 1\end{aligned}\right\}, \quad \left.\begin{aligned}(a - A)^2 + (b - B)^2 &= R^2\\ (a' - A)^2 + (b' - B)^2 &= R^2\end{aligned}\right\}.$$

From the first two of these we easily get

$$a'' = a - b\,.\frac{a - a'}{b - b'},$$

and from the other pair, we have, by expansion, &c.

$$a^2 - a'^2 - 2A\,.(a - a') + b^2 - b'^2 - 2B\,.(b - b') = 0;$$

$$\therefore\ \frac{a - a'}{b - b'}.(a + a') - 2A\,.\frac{a - a'}{b - b'} + b + b' - 2B = 0;$$

$$\therefore\qquad \frac{a - a'}{b - b'} = -\frac{b + b' - 2B}{a + a' - 2A};$$

$$\therefore\qquad a'' = a + b\,.\frac{b + b' - 2B}{a + a' - 2A}$$

$$= \frac{a^2 + b^2 + aa' + bb' - 2(Aa + Bb)}{a + a' - 2A}.$$

Similarly $b'' = \dfrac{a^2 + b^2 + aa' + bb' - 2(Aa + Bb)}{b + b' - 2B}$,

which give the equation.

Let the points coincide; then $a' = a$, $b' = b$;

$$a'' = \frac{a^2 + b^2 - (\text{A}a + \text{B}b)}{a - \text{A}} = \frac{a(a - \text{A}) + b(b - \text{B})}{a - \text{A}}$$

$$= a + b \cdot \frac{b - \text{B}}{a - \text{A}},$$

and $$b'' = b + a \cdot \frac{a - \text{A}}{b - \text{B}};$$

∴ the equation required is

$$\frac{x}{a + b \cdot \frac{b - \text{B}}{a - \text{A}}} + \frac{y}{b + a \cdot \frac{a - \text{A}}{b - \text{B}}} = 1.$$

Otherwise. (Fig. 19.)

Let A B be a tangent at any point P of the circle whose centre is C, the common ordinates of the tangent being O A, O B; then its equation is

$$\frac{x}{\text{O A}} + \frac{y}{\text{O B}} = 1.$$

$$\left.\begin{array}{ll} \text{But} & \text{M A} : \text{M P} :: \text{P}m : \text{C}m, \\ \text{and} & \text{N B} : \text{N P} :: \text{C}m : \text{P}m, \end{array}\right\}$$

$$\text{or} \quad \left.\begin{array}{l} \text{OA} - a : b :: b - \text{B} : a - \text{A} \\ \text{OB} - b : a :: a - \text{A} : b - \text{B}; \end{array}\right\}$$

$$\therefore \quad \text{O A} = a + b \cdot \frac{b - \text{B}}{a - \text{A}}, \text{O B} = b + a \cdot \frac{a - \text{A}}{b - \text{B}},$$

the same as before.

71. Prob.—*To find the equation of the normal at any point* (a, b) *of the circle* $(x - \text{A})^2 + (y - \text{B})^2 = \text{R}^2$.

Let $\left.\begin{array}{l}\dfrac{x}{a'} + \dfrac{y}{b'} = 1 \\[2ex] \dfrac{x}{a''} + \dfrac{y}{b''} = 1\end{array}\right\}$ be the equations of the

tangent and normal at that point; then $\because$ the normal passes through (a, b) at right angles to the tangent. (Ex. 2, p. 49.)

$$a'' = a - b \cdot \frac{b'}{a'}, \; b'' = b - a \cdot \frac{a'}{b'}.$$

But $\quad a' = a + b \cdot \dfrac{b - \mathrm{B}}{a - \mathrm{A}}, \; b' = b + a \cdot \dfrac{a - \mathrm{A}}{b - \mathrm{A}};$

$\therefore$ the equation required is

$$\frac{x}{a - b \cdot \dfrac{a - \mathrm{A}}{b - \mathrm{B}}} + \frac{y}{b - a \cdot \dfrac{b - \mathrm{B}}{a - \mathrm{A}}} = 1;$$

or it is

$$\frac{x}{\left(\dfrac{\mathrm{A}\,b - a\,\mathrm{B}}{b - \mathrm{B}}\right)} - \frac{y}{\left(\dfrac{\mathrm{A}\,b - a\,\mathrm{B}}{a - \mathrm{A}}\right)} = 1;$$

but the former is the more symmetrical, being the same as that for the tangent, with the exception of the *minus* sign instead of *plus* in the common ordinates, and the reciprocal of the differences of the co-ordinates of the given point and centre.

OTHERWISE. (Fig. 19.)

If N′ N″, the normal at P, have the common ordinates O A′, O B′, then the equation is

$$\frac{x}{\mathrm{O\,A'}} + \frac{y}{\mathrm{O\,B'}} = 1;$$

but

$$\left.\begin{array}{l} MA' : MP :: mC : mP \\ NB' : NP :: mP : Cm \end{array}\right\} \text{or} \left.\begin{array}{l} -OA'+a : b :: a-A : b-B \\ b-OB' : a :: b-B : a-A \end{array}\right\}$$

$$\therefore \quad OA' = a - b \cdot \frac{a-A}{b-B}, \; OB' = b - a \cdot \frac{b-B}{a-A};$$

the same as before.

72. Prob.—*To find the equation of a right line which passes through a given point* (a, b) *and touches a given circle*, viz.

$$(x-a')^2 + (y-b')^2 = R^2.$$

Assume the equation required to be

$$\frac{x}{a''} + \frac{y}{b''} = 1.$$

Since it passes through the given point (a, b), we have

$$\frac{a}{a''} + \frac{b}{b''} = 1 \; . \; . \; . \; . \; . \; (1).$$

Also, at the point of contact, the equations

$$\left.\begin{array}{ll} & \frac{x}{a''} + \frac{y}{b''} = 1 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (2) \\ \text{and} & (x-a')^2 + (y-b')^2 = R^2 \; . \; . \; . \; (3) \end{array}\right\}$$

are simultaneous.

And we have now only three equations involving the *four* unknown quantities, x, y, a'', b''. To obtain another equation we must adopt the condition of contact, viz. that the angle between the required tangent and the radius of the circle passing through the point of contact is a right angle.

Now the distances between the points (a, b), (a', b'); and between (a, b), (x, y), are

$\sqrt{\{(a-a')^2+(b-b')^2\}}$ and $\sqrt{\{(x-a)^2+(y-b)^2\}}$, and since the latter and R are the legs of a right angled $\triangle$ of which the other is the hypothenuse, we have

$$(x-a)^2+(y-b)^2+R^2=(a-a')^2+(b-b')^2 \quad \dots . \quad (4).$$

Equations (4) and (3) become, by reduction,

$$x^2+y^2-2ax-2by=a'^2+b'^2-2aa'-2bb'-R^2$$

$$x^2+y^2-2a'x-2b'y=-a'^2-b'^2+R^2;$$

$$\therefore\ (a'-a)x+(b'-b)y=a'^2+b'^2-R^2 \ \dots\dots (5).$$

From equations (5) and (3) we can easily find two values of x' and two of y', which being substituted in (2) will give two equations of the form

$$\frac{A}{a''}+\frac{B}{b''}=1,$$

and each of these being simultaneous with equation (1) will give two pairs of simultaneous values of

$$\frac{1}{a''} \text{ and } \frac{1}{b''};$$

whence we shall have the equations of two right lines which pass through the given point and touch the circle.

Problems similar to this are the following, which we propose as exercises to the student when he shall read this work a second time:—

73. Prob.—*To find the equation of the right line which touches two given circles.*

74. Prob.—*To find the equation of the circle which touches a given right line, and passes through a given point.*

75. Prob.—*To find the equation of the circle which*

touches two given circles and passes through a given point.

76. PROB.—*To find the equation of the circle which touches three given right lines.*

Problems of this kind, however, are much more easily resolved by common geometry than by analytical processes. For the general problem of contacts see Wright's 'Self-Examinations in Euclid.'

77. PROB.—*To find the area of a circle whose radius is* R.

If a regular polygon of n sides be inscribed in the circle, its area is found by dividing it into n equal $\triangle$s, whose bases are the sides of the polygon and vertices at the centre. The area of each of these $\triangle$s is

$$\frac{R^2}{2} \cdot \sin. \frac{360^\circ}{n}.$$

Hence the area of the polygon is

$$\frac{n R^2}{2} \cdot \sin. \frac{360^\circ}{n}.$$

But generally

$$2 \sin.^2 A = 1 - \cos. 2 A;$$

$$\therefore \quad 2 \sin.^2 \frac{360^\circ}{2^2 . 3} = 1 - \cos. 60 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$2 \sin.^2 \frac{360^\circ}{2^3 . 3} = 1 - \cos. 30^\circ = 1 - \frac{\sqrt{3}}{2} = \frac{2 - \sqrt{3}}{2};$$

$$\therefore \quad 2 \sin. \frac{360^\circ}{2^3 . 3} = \sqrt{(2 - \sqrt{3})}$$

$$= \sqrt{\frac{4+1}{2}} - \sqrt{\frac{4-1}{2}} = \sqrt{\frac{5}{2}} - \sqrt{\frac{3}{2}},$$

which may be reduced to its decimal value.

Proceeding thus continually it will at length be computed that

$$\sin.\frac{360^\circ}{2^{13}\cdot 3} = \cdot 0002556634;$$

$\therefore$ the area of the polygon of $2^{13}\cdot 3$ or 24576 sides is

$$12288 \,.\, R^2 \times \cdot 0002556634,$$

or $$3\cdot 141592 \times R^2 \text{ nearly};$$

which is always written

$$\pi \,.\, R^2.$$

78. Cor. 1.—Hence the area of a circle, whose radius is 1, is π or $3\cdot 14159$.

79. Cor. 2.—If s be one of the 24576 sides, and p the perpendicular let fall from the centre upon it; then

$$\text{the area of the circle} = s\,.\,\frac{p}{2} \times n = n\,s \times \frac{p}{2};$$

$$\therefore \text{circumference of} \odot \text{ rad. } R = n\,s = \frac{2 \text{ area}}{p} = \frac{2\pi R^2}{R}$$
$$= 2\,\pi\, R \text{ nearly}.$$

80. Cor. 3.—In the same way also if S be the length of the arc of the sector of a circle, its area is

$$\frac{R\,.\,S}{2}.$$

Section IV.

THEORY OF THE PARABOLA IN A CO-ORDINATE PLANE.

81. Def.—*A* Parabola *is a curve whose points are all in the same plane, and such that the distance of each*

from a given point is equal to the perpendicular drawn from that point upon a given right line.

82. Def.—*The given point is called the* Focus *of the Parabola, and the given right line is termed the* Directrix *of the Parabola.*

83. Def.—*The* Vertex *of a Parabola is the common point of the parabola, and the* ⊥ *drawn from the Focus upon the Directrix.*

84. Def.—*The* focal distance *of a parabola is the distance of the focus from the vertex; and the axis is that line produced.*

85. Prop.—*To find the rectangular equation to a parabola when the origin of co-ordinates is at the vertex and the axis of* x *is the focal distance indefinitely produced.* (Fig. 20.)

Let A be the vertex, S the focus of the parabola; A X, A Y the co-ordinate axes, and B N the directrix. Take P any point (x, y) in the parabola and draw P M parallel to A Y and P N ⊥ B N.

Then $SP = PN = MB = AM + AB$,

and making A S or A B $=$ S, for they are equal by the definition of the curve

$$SP = S + x.$$

$$\text{But}\quad SP^2 = y^2 + SM^2 = y^2 + (AM - AS)^2$$
$$= y^2 + (x - S)^2;$$

$$\therefore\quad y^2 + (x - S)^2 = (S + x)^2;$$

$$\therefore\quad y^2 + x^2 - 2Sx + a^2 = S^2 + 2Sx + x^2;$$

$$\therefore \quad y^2 = 4\,S\,x,$$

which is the equation required.

86. Prop.—*To find the equation of a parabola when the focus is the point* (a, b) *and the directrix the right line* $y = \sqrt{(a^2 + b^2)}$ *; that is, when the origin is in the curve and the directrix parallel to the axis of* x. (Fig. 21.)

Let P be any point in the parabola whose focus S is (a, b) and directrix B N $y = \sqrt{(a^2 + b^2)}$.

Then, by definition of the curve

$$SP = PN$$

P being any point (x, y).

But

$$S\,P^2 = (a - x)^2 + (y - b)^2, \text{ also } = (O\,B - P\,M)^2$$
$$= \{\sqrt{(a^2 + b^2)} - y\}^2;$$

$$\therefore \quad x^2 - 2\,a\,x + 2\{\sqrt{(a^2 + b^2)} - b\}\,y = 0,$$

the equation required.

This is the form of the equation of the parabola, which results from the theory of projectiles in a non-resisting medium.

87. Prop.—*Generally given the Focus of a Parabola, viz. the point* (a, b) *and the Directrix, viz. the right line whose equation is* $\frac{x}{a'} + \frac{y}{b'} = 1$, *to find the rectangular equation of the Parabola.* (Fig. 22.)

Let (x, y) be any point P in the parabola, the focus and directrix of which are S and C D. Join P S and draw P N $\perp$ C D, and P M parallel to the axis of y, meeting the axis of x in M. Draw also S m parallel to O X.

Then $\qquad SP^2 = Sm^2 + Pm^2$
$$= (x - a)^2 + (y - b)^2.$$

Now since PN is drawn from $(x, y) \perp$ to the right line

$$\frac{x}{a'} + \frac{y}{b'} = 1$$

$$PN = \frac{a'y + xb' - a'b'}{\sqrt{(a'^2 + b'^2)}}; \quad \text{(See 51.)}$$

$\therefore$ since by the definition $PS = PN$;

$$\therefore \quad (x - a)^2 + (y - b)^2 = \frac{(a'y + xb' - a'b')^2}{a'^2 + b'^2};$$

$$\therefore \ (x^2+y^2)(a'^2+b'^2) - 2a(a'^2+b'^2)x - 2b(a'^2+b'^2)y$$
$$+ (a^2 + b^2)(a'^2 + b'^2) = a'^2 y^2 + b'^2 x^2$$
$$+ 2a'b'xy - 2a'^2 b'y - a'b'^2 x + a'^2 b'^2;$$

$$\therefore \ a'^2 x^2 + b'^2 y^2 - 2a'b'xy - 2(aa'^2 + ab'^2 - a'b'^2)x$$
$$-2(bb'^2 + ba'^2 - b'a'^2)y = (a^2+b^2)(a'^2+b'^2) - a'^2 b'^2 \quad (1)$$

the equation required.

The form of this equation is

$$\left(\frac{x}{A} + \frac{y}{B}\right)^2 + \frac{x}{C} + \frac{y}{D} = 1.$$

$$\left.\begin{aligned}
\frac{(a^2 + b^2)(a'^2 + b'^2) - a'^2 . b'^2}{a'^2} &= A^2 \\
\frac{(a^2 + b^2)(a'^2 + b'^2) - a'^2 . b'^2}{b'^2} &= B^2 \\
\frac{(a^2 + b^2)(a'^2 + b'^2) - a'^2 b'^2}{aa'^2 + ab'^2 - a'b'^2} &= -2C \\
\frac{(a^2 + b^2)(a'^2 + b'^2) - a'^2 b'^2}{bb'^2 + ba'^2 - b'a'^2} &= -2D
\end{aligned}\right\} \quad (2).$$

88. Cor. 1.—When the directrix is parallel to the

axis of y, $b' = \infty$ and the equation becomes (dividing each term by b'^2)

$$y^2 - 2(a - a')x - 2by = a^2 + b^2 - a'^2 \ldots\ldots (3),$$

which is of the form

$$\frac{y^2}{B^2} + \frac{x}{C} + \frac{y}{D} = 1.$$

89. Cor. 2.—When the directrix is parallel to the axis of y, and the axis of y passes through the vertex, then $b' = \infty$ and $a' = -a$, and the equation becomes

$$y^2 - 4ax - 2by = b^2 \ldots\ldots (4).$$

90. Cor. 3.—When the directrix is parallel to the axis of y, and the axes of x and y both pass through the vertex, then $b' = 0$, $a' = -a$ and $b = 0$, and the equation becomes

$$y^2 = 4ax \ldots\ldots (5),$$

as in Art. 85.

From these equations it were easy to find a, b; a' b' and consequently the focus (a, b) and directrix $\frac{x}{a'} + \frac{y}{b'} = 1$ of any parabola belonging to any proposed equation of the above form. But the process, although easy enough in theory, is operose in practice.

91. Prop.—*To find the polar equation of the parabola whose focus is the pole, and focal distance the axis from which the traced angle is measured.* (Fig. 23).

The rectangular equation when the origin is at the vertex and axis of x the focal distance produced is

$$y^2 = 4Sx$$

S being the focal distance.

But $\quad y = r \sin. \theta,$

and $\quad x = \mathrm{AM} = \mathrm{AS} \pm \mathrm{SM} = \mathrm{S} - r \cos. \theta;$

$$\therefore \quad r^2 \sin.^2 \theta = 4\,\mathrm{S}\,(\mathrm{S} - r \cos. \theta);$$

$$r^2 + \frac{4\,\mathrm{S} \cos. \theta}{\sin.^2 \theta}\, r = \frac{4\,\mathrm{S}^2}{\sin.^2 \theta}$$

and solving this quadratic

$$r = -\frac{2\,\mathrm{S} \cos. \theta}{\sin.^2 \theta} \pm \sqrt{\left(\frac{4\,\mathrm{S}^2 \sin.^2 \theta}{\sin.^4 \theta} + \frac{4\,\mathrm{S}^2 \cos.^2 \theta}{\sin.^4 \theta}\right)}$$

$$= -\frac{2\,\mathrm{S} \cos. \theta \pm 2\,\mathrm{S}}{\sin.^2 \theta} = \frac{2\,\mathrm{S}}{1 - \cos.^2 \theta}(\pm 1 - \cos. \theta)$$

$$= \frac{2\,\mathrm{S}}{1 + \cos. \theta} \text{ or } -\frac{2\,\mathrm{S}}{1 - \cos. \theta} \quad \ldots\ldots \quad (6)$$

The former value belongs to P, the latter P′.

Otherwise, from the Definition.

Let B N be the directrix, &c.

$$\cos. \theta = -\cos. \mathrm{PSM} = -\frac{\mathrm{SM}}{\mathrm{SP}} = -\frac{\mathrm{BM} - 2\mathrm{AS}}{\mathrm{SP}}$$

$$= -\frac{\mathrm{SP} - 2\mathrm{S}}{\mathrm{SP}} = -1 + \frac{2\,\mathrm{S}}{\mathrm{SP}}$$

$$= -1 + \frac{2\,\mathrm{S}}{r}, \qquad \therefore\ r = \frac{2\,\mathrm{S}}{1 + \cos. \theta}$$

the same as before.

If the origin of polar co-ordinates be that of rectangular co-ordinates and the traced angle measured from the axis of x; then the general polar equation of the parabola would be obtained by substituting $r \cos. \theta$, $r \sin. \theta$ for x and y in the general rectangular equation.

92. Prop.—*To find the polar equation to the parabola, when the pole is at the focus, but the right line from which the traced angle is measured makes with the focal distance a given angle* α. (Fig. 23.)

Let S C be the straight line from which the traced angle is measured, S A the focal distance, &c. Then

$$\cos.\,(\alpha + \theta) = -\cos.\,\mathrm{PSM}$$

$$= -\frac{\mathrm{SM}}{\mathrm{SP}}$$

$$= -\frac{\mathrm{BM} - \mathrm{SB}}{\mathrm{SP}}$$

$$= -\frac{\mathrm{SP} - 2\mathrm{S}}{\mathrm{SP}} = -1 + \frac{2\mathrm{S}}{r}$$

$$\therefore \quad r = \frac{2\mathrm{S}}{1 + \cos.\,(\alpha + \theta)} \quad \ldots\ldots \quad (7)$$

This is the equation most frequently used by writers on Physical Astronomy when they treat of the motions of comets, α being usually the angular distance of the comet's perihelion from the line of the nodes.

Such are the useful equations of the parabola both rectangular and polar; and also the general rectangular equation.

Recapitulation of the Equations of a Parabola.

93.—1. *If the focus be* (a, b) *and directrix* $\frac{x}{a'} + \frac{y}{b'} = 1$;

then the equation is

$$a'^2 x^2 + b'^2 y^2 - 2\,a' b'\,xy - 2\,(a\,a'^2 + a\,b'^2 - a' b'^2) \times x - 2(b\,b'^2 + b\,a'^2 - b' a'^2)\,y = (a^2 + b^2)\,(a'^2 + b'^2) - a'^2\,b'^2$$

which is of the form

$$\left(\frac{x}{\mathrm{A}} + \frac{y}{\mathrm{B}}\right)^2 + \frac{x}{\mathrm{C}} + \frac{y}{\mathrm{D}} = 1.$$

2. *If the focus be* (a, b) *and directrix parallel to the axis of* y, *or if it be* $\frac{x}{a'} = 1$,

$$y^2 - 2(a - a')x - 2by = a^2 + b^2 - a'^2$$

which is of the form

$$\frac{y^2}{A^2} + \frac{x}{C} + \frac{y}{D} = 1.$$

3. *If the focus be* (a, b), *the directrix parallel to the axis of* y, *and the axis of* y *pass through the vertex; then*

$$\frac{y^2}{b^2} - \frac{4a}{b^2}x - \frac{2}{b} \cdot y = 1.$$

4. *If the focus be* (S, o), *the directrix parallel to axis of* y, *and axes of* (x, y) *both pass through the vertex; then*

$$y^2 = 4\,S\,x$$

S *being the focal distance.*

5. *If the origin of polar co-ordinates be at the focus, and* θ *measured from the focal distance; then*

$$r = \frac{2\,S}{1 + \cos.\,\theta}.$$

6. *If the origin of polar co-ordinates be the focus, and* θ *be measured from a right line passing through the focus and making a given angle* α *with the focal distance; then*

$$r = \frac{2\,S}{1 + \cos.\,(\alpha + \theta)}.$$

PROBLEMS ON RIGHT LINES, CIRCLES, AND PARABOLAS.

94. PROB.—*Given the equation of a parabola to find its common points with either axis.*

The general equation of parabolas is of the form

$$\left(\frac{x}{A} + \frac{y}{B}\right)^2 + \frac{x}{C} + \frac{y}{D} = 1.$$

Let $x = o$, then

$$\frac{y^2}{B^2} + \frac{y}{D} = 1;$$

$$\therefore\ y^2 + \frac{B^2}{D}y = B^2;$$

$$\therefore\ y = -\frac{B^2}{2D} \pm \sqrt{\left(B^2 + \frac{B^4}{4D^2}\right)}$$

$$= \frac{-B^2 \pm B\sqrt{(4D^2 + B^2)}}{2D}.$$

$\therefore$ the common points with the axis of y are

$$\left\{o, \frac{-B^2 + B\sqrt{(4D^2 + B^2)}}{2D}\right\}$$

and $\left\{o, \frac{-B^2 - B\sqrt{(4D^2 + B^2)}}{2D}\right\}$

Similarly, by making $y = 0$, we shall find the common points with the axis of x.

EXAMPLE.

To find the common points of the parabola.

$$(x + y)^2 + \frac{x}{2} + \frac{y}{3} = 1$$

Those with the axis of x are

$$\left(\frac{-1 + \sqrt{17}}{4}, o\right), \left(\frac{-1 - \sqrt{17}}{4}, o\right)$$

and those with the axis of y are

$$\left(o, \frac{-1 + \sqrt{37}}{6}\right), \left(o, \frac{-1 - \sqrt{37}}{6}\right).$$

95. PROB.—*To find the common points of a right line* $\frac{x}{a} + \frac{y}{b} = 1$ *and a parabola* $y^2 = 4\,S\,x$.

At the common points the equations are simultaneous.

$$\left.\begin{array}{r} \frac{x}{a} + \frac{y}{b} = 1 \\ y^2 = 4\,Sx \end{array}\right\} \therefore \frac{y^2}{4\,S\,a} + \frac{y}{b} = 1;$$

$$\therefore \; y = \frac{-2\,S\,a \pm 2\sqrt{(a\,b^2\,S + a^2\,S^2)}}{b};$$

$$\therefore x = \frac{y^2}{4\,S} = \frac{a\,b^2 + 2\,a^2\,S \mp 2\,a\sqrt{(a\,b^2\,S + a^2\,S^2)}}{b^2};$$

which give two points such as were required.

96. PROB.—*To find the common points of a circle*

$$(x - a)^2 + (y - b)^2 = R^2$$

and a parabola

$$y^2 = 4\,S\,x.$$

The process is similar to that of the last problem, but the result will give, in certain cases, *four* common points; in others *none*, or *one*, or *two*, or *three*.

97. PROB.—*To find the common points of two parabolas; viz.*

$$y^2 = 4\,S\,x \text{ and } \left(\frac{x}{A} + \frac{y}{B}\right)^2 + \frac{x}{C} + \frac{y}{D} = 1.$$

We shall also find in this problem, *one, two, three, four*, or no common points according to conditions which will be obvious from the corresponding pairs of co-ordinates of the resulting points.

98. PROB.—*Given two radius vectors* $R_{\prime}$ R' *of a parabola, and the angle* α *between them, to find the polar*

equation to the parabola, the pole being the focus and origin of traced angles the focal distance.

Let the equation required be

$$r = \frac{2a}{1 + \cos.\theta}$$

in which a is to be determined;

then, by the question, if θ be supposed the traced angle corresponding to R, we have

$$\text{R} = \frac{2a}{1 + \cos.\theta} = \frac{a}{\cos.^2\frac{\theta}{2}},$$

$$\text{R}' = \frac{2a}{1 + \cos.(\theta + \alpha)} = \frac{a}{\cos.^2\left(\frac{\theta}{2} + \frac{\alpha}{2}\right)};$$

$$\therefore \frac{\text{R}}{\text{R}'} = \left\{\frac{\cos.\left(\frac{\theta}{2}+\frac{\alpha}{2}\right)}{\cos.\frac{\theta}{2}}\right\}^2 = \left\{\cos.\frac{\alpha}{2} - \sin.\frac{\alpha}{2}\tan.\frac{\theta}{2}\right\}^2;$$

$$\therefore \sin.\frac{\alpha}{2}\tan.\frac{\theta}{2} = \cos.\frac{\alpha}{2} \pm \sqrt{\frac{\text{R}}{\text{R}'}},$$

$$\text{and } \tan.\frac{\theta}{2} = \cot.\frac{\alpha}{2} \pm \sqrt{\frac{\text{R}}{\text{R}'}}.\operatorname{cosec}.\frac{\alpha}{2} \quad \ldots\ldots (1)$$

which determines θ. But to find a, we have

$$\frac{\text{R}}{a} = \frac{1}{\cos.^2\frac{\theta}{2}} = 1 + \tan.^2\frac{\theta}{2}$$

$$= 1 + \cot.^2\frac{\alpha}{2} \pm 2\sqrt{\frac{\text{R}}{\text{R}'}}.\cot.\frac{\alpha}{2}\operatorname{cosec}.\frac{\alpha}{2}$$

$$+ \frac{\text{R}}{\text{R}'}\operatorname{cosec}.^2\frac{\alpha}{2},$$

$$= \text{cosec.}^2 \frac{\alpha}{2}\left(1 \pm 2\sqrt{\frac{R}{R'}} \cdot \cos. \frac{\alpha}{2} + \frac{R}{R'}\right);$$

$$\therefore\ a = \frac{R \sin.^2 \frac{\alpha}{2}}{1 \pm 2\sqrt{\frac{R}{R'}} \cdot \cos. \frac{\alpha}{2} + \frac{R}{R'}};$$

$\therefore$ the equation required is

$$r = \frac{2\,R\,R' \sin.^2 \frac{\alpha}{2}}{R + R' \pm 2\sqrt{R\,R'} \cdot \cos. \frac{\alpha}{2}} \times \frac{1}{1 + \cos.\ \theta}.$$

This is a problem in the theory of Comets.

99. Prob.—*To find the equation of the Tangent at any point (a, b) of the parabola $y^2 = 4\,S\,x$.*

Let the equation of the secant passing through (a, b), (a', b') be

$$\frac{x}{a''} + \frac{y}{b''} = 1,$$

then
$$\left.\begin{array}{l} \dfrac{a}{a''} + \dfrac{b}{b''} = 1 \\[2ex] \dfrac{a'}{a''} + \dfrac{b'}{b''} = 1 \end{array}\right\} \quad \left.\begin{array}{l} b^2 = 4\,S\,a \\ b'^2 = 4\,S\,a' \end{array}\right\}$$

From the first two of these, we get

$$a'' = a - b \cdot \frac{a - a'}{b - b'},\ b'' = b - a \cdot \frac{b - b'}{a - a'}$$

and from the other pair

$$\frac{a - a'}{b - b'} = \frac{b + b'}{4\,S}$$

$$\therefore\ a'' = a - b \cdot \frac{b + b'}{4\,S};\ \text{and } b'' = b - a \cdot \frac{4\,S}{b + b'}$$

Let the point (a', b') coincide with (a, b); then $a = a'$, $b = b'$, and we get

$$\left.\begin{aligned} a'' &= a - \frac{b^2}{2\,S} \\ &= a - 2\,a \\ &= -\,a \end{aligned}\right\};\qquad \left.\begin{aligned} b'' &= b - \frac{2\,a\,S}{b} \\ &= b - \frac{b}{2} \\ &= \frac{b}{2} \end{aligned}\right\}$$

$\therefore$ the equation required is

$$\frac{x}{(-\,a)} + \frac{y}{\left(\frac{b}{2}\right)} = 1.$$

100. Prob.—*To find the perpendicular distance from the focus of a Parabola* $y^2 = 4\,S\,x$ *to the tangent at the point* (a, b).

Let (a', b') be the focus, and

$\frac{x}{a''} + \frac{y}{b''} = 1$, the equation of the tangent;

then the distance required is (see 51)

$$p = \left(\frac{a'}{a''} + \frac{b'}{b''} = 1\right) \cdot \frac{a''\,b''}{\sqrt{(a''^2 + b''^2)}}$$

But $a' = S$ and $b' = 0$;

also $a'' = -\,a$ and $b'' = \frac{b}{2}$

$$\therefore p = \left(\frac{S}{a} + 1\right) \cdot \frac{a\,b}{2\sqrt{\left(a^2 + \frac{b^2}{4}\right)}}$$

$$= \sqrt{\frac{(S + a)\,b}{2}} \cdot \frac{1}{\sqrt{a}\,\sqrt{(S + a)}}$$

$$= \sqrt{\frac{S + a}{a}} \cdot \sqrt{} \cdot S\,a = \sqrt{} S\,(S + a)$$

Now, if r be the distance of (a, b) from the focus; we have

$$r^2 = b^2 + (a - S)^2 = 4\,S\,a + a^2 - 2\,S\,a + S^2$$
$$= (S + a)^2,$$
$$r = S + a,$$

and $\therefore\ p^2 = S\,r$;

and since (a, b) is any point of the parabola, p and r are variable. This is called by several writers the equation between the perpendicular and radius vector. It is adopted in Newton's Principia, and many other works, but rarely by modern philosophers.

Hence, if P T be the ultimate position of the secant, that is, of the tangent; then A being the vertex, we have

A T = A M in length, but opposite in direction; which we also learn from Geometrical Treatises of Conic Sections.

101. Prob.—*To find the equation to the Normal at any point (a, b) of the parabola $y^2 = 4\,S\,x$.*

Let $\left.\begin{array}{l} \dfrac{x}{a'} + \dfrac{y}{b'} = 1 \\ \dfrac{x}{a''} + \dfrac{y}{b''} = 1 \end{array}\right\}$ be the equations of the tangent and normal respectively; then we have (Ex. 2. p. 40)

$$a'' = a - b\,\frac{b'}{a'},\ b'' = b - a\,.\,\frac{a'}{b'}\,. \qquad \therefore$$

But $a' = -a$, and $b' = \dfrac{b}{2}$,

$$\therefore \quad a'' = a + b \cdot \frac{b}{2a} = a + \frac{4Sa}{2a} = a + 2S$$

$$\text{and } b'' = b + \frac{2a^2}{b} = \frac{4Sa + 2a^2}{b} = \frac{2a}{b}(a + 2S),$$

$\therefore$ the equation required is

$$\frac{x}{a + 2S} + \frac{y}{\frac{2a}{b}(a + 2S)} = 1.$$

102. Prob.—*To find the equation of the Tangent at any point (a, b) of the general parabola*

$$\left(\frac{x}{A} + \frac{y}{B}\right)^2 + \frac{x}{C} + \frac{y}{D} + 1 = 0$$

Let $\frac{x}{a''} + \frac{y}{b''} = 1$ be the equation of the secant passing through the point (a, b), (a', b');

$$\text{then} \quad \left.\begin{aligned} \frac{a}{a''} + \frac{b}{b''} &= 1 \\ \frac{a'}{a''} + \frac{b'}{b''} &= 1 \end{aligned}\right\} \text{which give } a'' = a - b \cdot \frac{a - a'}{b - b'};$$

$$\text{Also,} \quad \left.\begin{aligned} \left(\frac{a}{A} + \frac{b}{B}\right)^2 + \frac{a}{C} + \frac{b}{D} + 1 &= 0 \\ \left(\frac{a'}{A} + \frac{b'}{B}\right)^2 + \frac{a'}{C} + \frac{b'}{D} + 1 &= 0 \end{aligned}\right\}$$

$$\therefore \quad \frac{a^2 - a'^2}{A^2} + \frac{2(a - a')}{A} + \frac{a - a'}{C} + \frac{b^2 - b'^2}{B^2}$$
$$+ \frac{2(b - b')}{B} + \frac{b - b'}{D} = 0,$$

$$\therefore \quad (a - a')\left(\frac{a + a'}{A^2} + \frac{2}{A} + \frac{1}{C}\right)$$

$$= -\left(\frac{b + b'}{B^2} + \frac{2}{B} + \frac{1}{D}\right)(b - b');$$

$$\therefore \quad \frac{a - a'}{b - b'} = -\frac{\dfrac{b + b'}{B^2} + \dfrac{2}{B} + \dfrac{1}{D}}{\dfrac{a + a'}{A^2} + \dfrac{2}{A} + \dfrac{1}{C}};$$

$$\therefore \quad a'' = a + \frac{\dfrac{b + b'}{B^2} + \dfrac{2}{B} + \dfrac{1}{D}}{\dfrac{a + a'}{A^2} + \dfrac{2}{A} + \dfrac{1}{C}}.$$

Let the secant become the tangent; then $a' = a$, $b' = b$

and $$a'' = a + \frac{\dfrac{2\,b}{B^2} + \dfrac{2}{B} + \dfrac{1}{D}}{\dfrac{2\,a}{A^2} + \dfrac{2}{A} + \dfrac{1}{C}}$$

$$= \frac{2\,ABCD\,(a^2+b^2)+2\,ab\,CD\,(A^2+B^2)+A^2B^2\,(aC+bD)}{AC\,(2\,a\,BD + 2\,b\,AD + AB^2)}.$$

Similarly

$$b'' = \frac{2\,ABCD\,(a^2+b^2)+2\,abCD(A^2+B^2)+A^2B^2(aC+bD)}{BD\,(2\,a\,BC + 2\,b\,AC + A^2B)};$$

whence the equation required.

From this equation of the tangent, that of the normal whose common ordinates are a'', b'', will easily be found from the values

$$a'' = a - b.\frac{b'}{a'}$$

$$b'' = b - a.\frac{a'}{b'}$$

103. DEF.—*The* SECANT CIRCLE *of a curve at any point of it, is that which passes through that point and any two other points of the curve.*

That a circle may be described through any three points is evident from Euclid, B. iv., Prop. 5.

104. DEF.—*The* OSCULATING CIRCLE, *or* CIRCLE OF CURVATURE *at any point of a curve, is that circle which the secant circle becomes when those other two common points coincide with the given common point.*

105. DEF.—*The* RADIUS OF CURVATURE *is that of the circle of curvature or osculating circle.*

106. DEF.—*The* CENTRE OF CURVATURE *is that of the circle of curvature or osculating circle.*

107, DEF.—*The* EVOLUTE OF A CURVE *is the locus of the centres of curvature of every point of the curve.*

108. DEF.—*The* INVOLUTE OF A GIVEN CURVE *is that curve whose evolute is the given curve.*

109. PROB.—*To find the equation of the Osculating Circle at any point (a, b) of a parabola* $y^2 = 4\,S\,x$.

Supposing (A, B) the centre of the secant circle passing through the three points (a, b), (a', b') and (a'', b'') of the parabola, and ρ its radius, then the equation of the secant circle is

$$(x - A)^2 + (y - B)^2 = \rho^2,$$

and it only remains to determine A, B and ρ on the supposition that the points (a', b'), (a'', b'') coincide with the point (a, b).

Now, since the secant circle passes through (a, b), (a', b'), (a'', b'') we have

$$\left.\begin{aligned}(a - A)^2 + (b - B)^2 &= \rho^2\\ (a' - A)^2 + (b' - B)^2 &= \rho^2\\ (a'' - A)^2 + (b'' - B)^2 &= \rho^2\end{aligned}\right\}$$

which by subtraction or eliminating ρ give

$$\left.\begin{aligned}2A(a - a') + 2B(b - b') &= a^2 - a'^2 + b^2 - b'^2\\ \text{and } 2A(a - a'') + 2B(b - b'') &= a^2 - a''^2 + b^2 - b''^2\end{aligned}\right\}$$

But since (a, b), (a', b') (a'', b'') are points of the parabola $y^2 = 4Sx$,

$$\therefore \left.\begin{aligned}b^2 &= 4Sa\\ b'^2 &= 4Sa'\\ b''^2 &= 4Sa''\end{aligned}\right\}, \quad \therefore \left.\begin{aligned}b^2 - b'^2 &= 4S(a - a')\\ b^2 - b''^2 &= 4S(a - a'')\\ b'^2 - b''^2 &= 4S(a' - a'')\end{aligned}\right\};$$

$$\left.\begin{aligned}\therefore\ 2A(a - a') + 2B \,.\, 4S \,.\, \frac{a - a'}{b + b'} &= a^2 - a'^2\\ &+ 4S\,.\,(a - a')\\ 2A(a - a'') + 2B \,.\, 4S \,.\, \frac{a - a''}{b + b''} &= a^2 - a''^2\\ &+ 4S(a - a'')\end{aligned}\right\};$$

$$\left.\begin{aligned}\text{or, } 2A + \frac{8SB}{b + b'} &= a + a' + 4S\\ \text{and } 2A + \frac{8SB}{b + b''} &= a + a'' + 4S\end{aligned}\right\};$$

$$\therefore\ 8S\,.\,B\,.\,\frac{b'' - b'}{(b + b')(b + b'')} = -(a'' - a');$$

$$\therefore\ B = -\frac{a'' - a'}{b'' - b'} \,.\, \frac{(b + b')(b + b'')}{8S}$$

$$= -\frac{b' + b''}{4S} \,.\, \frac{(b + b')(b + b'')}{8S}.$$

$$\left.\begin{aligned}\text{or, } B &= -\frac{(b + b')\,(b + b'')\,(b' + b'')}{32\,S^2} \quad \ldots\ldots (1)\\ \text{hence } A &= \frac{a + a'}{2} + 2\,S + \frac{(b + b'')\,(b' + b'')}{8\,S} \ldots (2)\end{aligned}\right\}.$$

These values of A and B being substituted in the equation

$$(a - A)^2 + (b - B)^2 = \rho^2$$

will give ρ; and A, B and ρ being determined, we have found the equation of the secant circle passing through (a, b), (a', b'), (a'', b''). Now, let this secant circle become an osculating circle, or circle of curvature; that is, let (a', b'), (a'', b'') coincide with (a, b), and we get

$$A = a + 2\,S + \frac{4\,b^2}{8\,S} = a + 2\,S + \frac{16\,.\,S\,a}{8\,S}$$

$$= 2\,S + 3\,a$$

$$B = -\frac{8\,b^3}{32\,S^2} = -\frac{b^3}{4\,S^2} = -\frac{b\,.\,4\,S\,a}{4\,S^2} = -\frac{ab}{S},$$

and $\rho^2 = (a - A)^2 + (b - B)^2$

$$= (2\,S + 2\,a)^2 + \left(b + \frac{b^3}{4\,S^2}\right)^2$$

$$= 4\,(S + a)^2 + b^2\left(1 + \frac{4\,S\,a}{4\,S^2}\right)^2$$

$$= 4\,(S + a)^2 + 4\,S\,a\left(1 + \frac{a}{S}\right)^2$$

$$= 4\,(S + a)^2\left(1 + \frac{a}{S}\right) = 4\,.\,\frac{(S + a)^3}{S};$$

$\therefore$ the osculating radius or radius of curvature is

$$\rho = 2\,.\,(S + a)\sqrt{\frac{S + a}{S}};$$

and the equation of the osculating circle is

$$\{x - (2\,S + 3\,a)\}^2 + \left\{y + \frac{a\,b}{S}\right\}^2 = 4\,.\,\frac{(S + a)^3}{S}.$$

110. Prob.—*To find the equation of the Evolute of a parabola* $y^2 = 4\,S\,x$.

If (a, b) be any point of the parabola, and (x, y) the corresponding point of the evolute; then by the last problem

$$\left.\begin{aligned} a' &= 2\,S + 3\,a \\ b' &= -\frac{a\,b}{S} \end{aligned}\right\};$$

and by the equation of the parabola

$$b^2 = 4\,S\,a\,.$$

Eliminating a, b from these three equations, we get an equation between a' and b' which will be the equation required. To effect this object we first get

$$b'^2 = \frac{a^2}{S^2}\,.\,b^2 = 4\,\frac{a^3}{S} = \frac{4}{S}\,.\left(\frac{a' - 2\,S}{3}\right)^3;$$

and since (a, b) is any point of the parabola, the corresponding point (a', b') is any point of its evolute. Hence the equation of the evolute of a parabola $y^2 = 4\,S\,x$ is

$$y^2 = \frac{4}{S}\cdot\left(\frac{x - 2\,S}{3}\right)^3$$

$$\text{or}\quad \left(\frac{27}{4}\,S\,y^2\right)^{\frac{1}{3}} = x - 2\,S$$

$$\therefore\quad x - 3\,\sqrt[3]{\frac{S}{4}}\,.\,y^{\frac{2}{3}} = 2\,S$$

$$\text{or}\,\frac{x}{2\,S} - \frac{3}{2}\,.\,\frac{(2\,S)^{\frac{1}{3}}}{2\,S}\,.\,y^{\frac{2}{3}} = 1$$

$$\text{or} \quad \frac{x}{2\,S} - \frac{3}{2} \cdot \left(\frac{y}{2\,S}\right)^{\frac{2}{3}} = 1\,.$$

which, being the equation of no curve that is useful in Natural Philosophy, is a matter of curiosity only.

We shall find that the *evolutes* of the other conic sections are equally useless, and indeed those of all other curves whatsoever. But we shall investigate their equations in every case, merely to show the student the possibility and method of all such processes. These are all usually performed by the rules of Fluxions or the Differential Calculus—rules which will hence appear to be superseded by common Algebra.

111. Prob.—*To find the area of a parabola* $y^2 = 4\,S\,x$ *between the points* (o, o), (a, b). (Fig. 24.)

Let the co-ordinate b be divided into n parts each $= k$, and the corresponding parts of a be h, h', h'', h''', &c., then, as in figure, the sum of the rectangles in the convex part, is

$$h\,k + h'\,k + h''\,k + \ldots\ldots\ldots h^{(n)}\,k$$

$$\text{But } h = \frac{k^2}{4\,S}\,,\; h' = \frac{2^2\,k^2}{4\,S}\,,\; h'' = \frac{3^2\,k^2}{4\,S}\,,\ \&c.,$$

$$\therefore \quad \text{sum} = \frac{k^3}{4\,S} \cdot (1 + 2^2 + 3^2 + \ldots\ldots n^2)$$

$$= \frac{k^3}{4\,S} \times \frac{n\,(n+1)\,(2\,n+1)}{1\,.\,2\,.\,3} = \frac{b^3}{4\,S\,n^3} \cdot \frac{n\,(n+1)\,(2\,n+1)}{1\,.\,2\,.\,3}.$$

Let n be infinite; then

$$\text{Area A P N} = \frac{b^3}{4\,S} \times \frac{1}{3} = \frac{b^2}{4\,S} \cdot \frac{b}{3} = \frac{1}{3} \cdot a\,b$$

$$\therefore \text{ area APM} = \frac{2}{3}\,a\,b\,.$$

212. PROB.—*To find the sectorial area of a parabola* $r = \frac{2S}{1+\cos.\theta}$ *or* $= S . \sec.^2 \frac{\theta}{2}$ *which is comprised between the radius vectors* R, R′, *whose pole is the focus.* (Fig. 25.)

Let A S P be the parabola, A being its vertex and S its focus; and suppose P′, P the points (R, α), R′, α'); then completing the figure, we have

$$\begin{aligned}
\text{Area } PSP' &= ASP' - ASP \\
&= AM'P' - SM'P' - AMP + SMP \\
&= \tfrac{2}{3}(AS + SM')\,P'M' - \tfrac{1}{2}SM'.P'M' \\
&\quad - \tfrac{2}{3}(AS + SM)\,PM + \tfrac{1}{2}SM.PM \\
&= \tfrac{2}{3}AS.(P'M' - PM) + \tfrac{1}{6}.(SM'.PM' \\
&\qquad\qquad - SM.PM) \\
&= \tfrac{2}{3}S.(R'\sin.\alpha' - R\sin.\alpha) \\
&\quad - \tfrac{1}{6}(R'^2\sin.\alpha'\cos.\alpha' - R^2\sin.\alpha\cos.\alpha)
\end{aligned}$$

$$\text{But } R = \frac{S}{\cos.^2\frac{1}{2}\alpha},\ R' = \frac{S}{\cos.^2\frac{1}{2}\alpha'},$$

$$\sin.\alpha = 2\sin.\tfrac{1}{2}\alpha,\ \cos.\tfrac{1}{2}\alpha, \text{\&c.}$$

$$\begin{aligned}
\therefore\ \frac{3\text{ area }PSP'}{S^2} &= 4(\tan.\tfrac{1}{2}\alpha' - \tan.\tfrac{1}{2}\alpha) \\
&\quad - \tan.\tfrac{1}{2}\alpha'.\left(\frac{\cos.\alpha'}{\cos.^2\frac{1}{2}\alpha'} - \tan.\tfrac{1}{2}\alpha\frac{\cos.\alpha}{\cos.^2\frac{1}{2}\alpha}\right) \\
&= 4(\tan.\tfrac{1}{2}\alpha' - \tan.\tfrac{1}{2}\alpha) \\
&\quad - \tan.\tfrac{1}{2}\alpha'(1 - \tan.^2\tfrac{1}{2}\alpha) + \tan.\tfrac{1}{2}\alpha(1 - \tan.^2\tfrac{1}{2}\alpha) \\
&= 3(\tan.\tfrac{1}{2}\alpha' - \tan.\tfrac{1}{2}\alpha) + \tan.^3\tfrac{1}{2}\alpha' - \tan.^3\tfrac{1}{2}\alpha \\
&= (\tan.\tfrac{1}{2}\alpha' - \tan.\tfrac{1}{2}\alpha)(3 + \tan.^2\tfrac{1}{2}\alpha' \\
&\qquad + \tan.\tfrac{1}{2}\alpha'.\tan.\tfrac{1}{2}\alpha + \tan.^2\tfrac{1}{2}\alpha);
\end{aligned}$$

$$\therefore \quad \text{area } PSP = \frac{S^2}{3} \times (\tan.\tfrac{1}{2}\alpha' - \tan.\tfrac{1}{2}\alpha)\ \{\sec.^2\tfrac{1}{2}\alpha'$$
$$+ \sec.^2\tfrac{1}{2}\alpha + 1 + \tan.\tfrac{1}{2}\alpha' . \tan.\alpha\}$$

which is expressed in terms of the traced angles, divested of radius vectors.

It may, however, be more elegantly expressed in terms of the chord c joining P, P′ and R, R′.

$$\text{First } \sec.^2\tfrac{1}{2}\alpha' = \frac{R'}{S}, \sec.^2\tfrac{1}{2}\alpha = \frac{R}{S},$$

and

$$1 + \tan.\tfrac{1}{2}\alpha' . \tan.\tfrac{1}{2}\alpha = \cos.\tfrac{1}{2}(\alpha' - \alpha) \sec.\tfrac{1}{2}\alpha' . \sec.\tfrac{1}{2}\alpha$$
$$= \sqrt{\frac{1 + \cos.(\alpha' - \alpha)}{2}} . \frac{\sqrt{RR'}}{S}$$
$$= \sqrt{\frac{1 + \frac{R'^2 + R^2 - c^2}{2RR'}}{2}} . \frac{\sqrt{RR'}}{S}$$
$$= \frac{1}{2S} . \sqrt{(R'+R+c)}\sqrt{(R'+R-c)}$$

Also

$$(\tan.\tfrac{1}{2}\alpha' - \tan.\tfrac{1}{2}\alpha)^2 = \sec.^2\tfrac{1}{2}\alpha' + \sec.^2\tfrac{1}{2}\alpha$$
$$-2(1 + \tan.\tfrac{1}{2}\alpha' . \tan.\tfrac{1}{2}\alpha)$$
$$= \frac{R'+R}{S} - \frac{1}{S}\sqrt{(R'+R+c)} . \sqrt{(R'+R-c)}$$
$$= \frac{R'+R+c}{2S}$$
$$- \frac{1}{S}\sqrt{(R'+R+c)}\sqrt{(R'+R-c)} + \frac{R'+R-c)}{2S}$$

therefore

$$\tan.\tfrac{1}{2}\alpha' - \tan.\tfrac{1}{2}\alpha = \sqrt{\frac{R'+R+c}{2S}} - \sqrt{\frac{R'+R-c}{2S}}$$

$$\therefore \text{area PSP}' = \frac{S^2}{3}\left(\sqrt{\frac{R'+R+c}{2S}} - \sqrt{\frac{R'+R-c}{2S}}\right)$$

$$\times \left\{\frac{R'+R}{S} + \frac{1}{2S}\sqrt{(R+R'+c)}\,\sqrt{(R+R'-c)}\right\}$$

$$= \frac{\sqrt{S}}{6\sqrt{S}} \cdot \{\sqrt{(R'+R+c)} - \sqrt{(R'+R-c)}\}\{(R+R+c)$$

$$+ \sqrt{(R'+R+c)}\,\sqrt{(R'+R-c)} + (R'+R-c)\}$$

$$= \frac{\sqrt{S}}{6\sqrt{2}} \cdot \{(R'+R+c)^{\frac{3}{2}} - (R'+R-c)^{\frac{3}{2}}\};$$

which is the celebrated Theorem of LAMBERT, so useful in the Cometary Theory.

113. PROP.—*To trace the figure of the parabola*

$$y^2 = 4\,S\,x.$$

When $x = 0$, then also $y = 0$; $\therefore$ the curve passes through the origin of co-ordinates.

When x is negative, then y is imaginary, and $\therefore$ no part of the curve lies on the side of the negative axis of x.

When x is positive, and of any magnitude whatever, then $y = \pm\, 2\sqrt{S}\,.\,\sqrt{x}$ has two equal values on different sides of the axis of x. Consequently the parabola has two infinite branches symmetrical with respect to the axis of x.

Similarly the figure of any parabola

$$\left(\frac{x}{A} + \frac{y}{B}\right)^2 + \frac{x}{C} + \frac{y}{D} = 1$$

may be traced, by assuming several values of x and finding

the corresponding values of y. Having thus obtained a number of points in the figure, by joining them, we shall trace it.

Section V.

THEORY OF THE ELLIPSE IN A CO-ÖRDINATE PLANE.

114. Def.—*An* Ellipse *is a curve every point of which is in one plane, and such that the* sum *of the distances of that point from two given points in that plane is constant.* (Fig. 26.)

If S and H be the given or fixed points and P, P′ any two points in the ellipse; then will

$$SP + PH = SP' + P'H,$$

and so on for any other points; that is, the sum of the distances is always the same.

115. Def.—*The* Foci *of an ellipse are the two given points from which the sum of the distances of every point in the ellipse is the same.*

In the diagram S and H are the foci.

116. Prop.—*The equation of an ellipse, when the foci are the points* $(-E, o)$, (E, o), ($2E$ *being the distance between the foci*), *is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

in which a *and* b *are the semiaxes* O A O B. (Fig. 27.)

Let S, H be the foci, O the origin of co-ordinates, &c.

as in the figure; then P being any point (x, y) of the curve, we have,

$$SP + PH = 2a.$$

But $\left.\begin{array}{l} SP^2 = SM^2 + PM^2 = (E + x)^2 + y^2 \\ HP^2 = HM^2 + PM^2 = (E - x)^2 + y^2 \end{array}\right\}$

$\therefore \quad SP^2 - HP^2 = 4Ex = \text{also } (SP + PH)(SP - PH)$
$\qquad\qquad = 2a(2SP - 2a);$

$$\therefore \quad SP = \frac{Ex}{a} + a;$$

$$\therefore \quad SP^2 = \frac{E^2}{a^2}x^2 + 2Ex + a^2 = \text{also } E^2 - 2Ex + x^2 + y^2;$$

$$\therefore \quad x^2\left(1 - \frac{E^2}{a^2}\right) + y^2 = a^2 - E^2 = b^2;$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the equation required.

117. Prob.—*To trace the figure of an Ellipse* (Fig. 27),

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and find its maxima and minima co-ordinates.

Since $x = \pm \frac{a}{b} \sqrt{(b^2 - y^2)}$, y cannot be $> \pm b$ and then $x = 0$;

$\therefore$ at the points $(o, \pm b)$ the positive and negative values of y are greatest.

Since $y = \pm \frac{b}{a} \sqrt{(a^2 - x^2)}$, the greatest positive and negative values of x are $\pm a$ and then $y = 0$;

$\therefore$ at the points $(\pm a, o)$ the values of x are greatest.

At all intermediate points to these, positive and negative values of x are always equal in pairs, as also are those of y. Consequently the ellipse is symmetrical on both sides the axis.

Any number of points being found by assuming values for one co-ordinate, those of the other may be found and the curve traced.

Mechanical Description.

118. Rule.—Fasten two pins in the foci S, H, to which attach a thread S P H the length of the constant distance of the foci from every point in the curve. Then, placing a pen or pencil so as always to keep the string stretched, let it be carried all round, and it will trace the curve with considerable accuracy.

119. Def.—The axis major is the line A A′ or $2a$; that is, twice the maximum value of x.

120. Def.—The axis minor is the line B B′ or $2b$; that is, twice the maximum value of y.

121. Def.—The vertices are the points A, A′.

122. Def.—The centre of an ellipse is the point which bisects the distance between the foci.

123. Prop.—*The equation of an ellipse whose foci are the points* $(a - E, o)$, $(a + E, o)$, 2 a *being the axis major and* 2 E *the distance between the foci, is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 \cdot \frac{x}{a},$$

in which $b^2 = a^2 - E^2$. (Fig. 28.)

Let S and H be the foci and P any point in the curve;

then O being the origin of co-ordinates, and C the bisection of S H, &c. we have

$$OC = a,\ SC = E \text{ and } BC = b.$$

Also $SP + PH = 2a$; for $SP + PH = SO + HO$
$= OC - SC + OC + SC = 2OC = 2a$;

and $SP^2 = SM^2 + PM^2 = (OM - OS)^2 + y^2$
$= \{x - (a - E)\}^2 + y^2$;

and $HP^2 = HM^2 + PM^2 = (OH - OM)^2 + y^2$
$= (a + E - x)^2 + y^2$;

$\therefore SP^2 - HP^2 = 4(x - a).E = (SP + HP)(SP - HP)$
$= 2a.(2SP - 2a)$;

$$\therefore \quad SP = a - E + \frac{E}{a}x;$$

$$\therefore \quad SP^2 = (a - E)^2 + 2(a - E).\frac{E}{a}x + \frac{E^2}{a^2}x^2;$$

$$\text{also} \quad = x^2 - 2(a - E)x + (a - E)^2 + y^2;$$

$$\therefore \quad x^2\left(1 - \frac{E^2}{a^2}\right) + y^2 = 2\frac{(a^2 - E^2)}{a}x;$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{a^2 - E^2} = 2\frac{x}{a}.$$

But $SB + HB = 2a = 2SB$, $\therefore SB = a$

and $a^2 - E^2 = SB^2 - SC^2 = BC^2 = b^2$;

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2.\frac{x}{a},$$

the equation required.

124. Prop.—*The equation of an ellipse whose foci are the points* (o, o), $(2E, o)$, $2E$ *being the right line joining the foci, is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 \cdot \frac{\sqrt{(a^2 - b^2)}}{a^2} x + \frac{b^2}{a^2}$$

where a *and* b *are the semi-axes.*

For S being the origin of co-ordinates, we have (Fig. 29.)

$$SP^2 = SM^2 + PM^2 = x^2 + y^2$$
$$HP^2 = HM^2 + PM^2 = (2E - x)^2 + y^2;$$
$$\therefore SP^2 - HP^2 = 4Ex - 4E^2 = (SP + HP)(SP - HP)$$
$$= 2a(2SP - 2a);$$

$$\therefore SP = a - \frac{E^2}{a} + \frac{Ex}{a} = \frac{a^2 - E^2}{a} + \frac{Ex}{a} = \frac{b^2}{a} + \frac{Ex}{a} \quad (123);$$

$$\therefore \quad SP^2 = \frac{b^4}{a^2} + 2\frac{b^2}{a^2}Ex + \frac{E^2x^2}{a^2} = \text{also } x^2 + y^2;$$

$$\therefore \quad x^2\left(1 - \frac{E^2}{a^2}\right) + y^2 = \frac{2b^2}{a^2}Ex + \frac{b^4}{a^2};$$

$$\therefore \quad \frac{b^2x^2}{a^2} + y^2 = \frac{2b^2\sqrt{(a^2 - b^2)}}{a^2}x + \frac{b^4}{a^2};$$

$$\text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2\sqrt{(a^2 - b^2)}}{a^2}x + \frac{b^2}{a^2},$$

the equation required.

125. Prop.—*Given the foci* (a, b), (a', b') *and* 2 A, *the constant sum of the distances of any point* (x, y) *in the ellipse from the foci, to find the equation of the ellipse.* (Fig. 30.)

If (x, y) be any point P in the ellipse, and S, H the foci (a, b), (a', b'); then

$$\left.\begin{aligned} SP^2 &= (x - a)^2 + (y - b)^2 \\ \text{and} \qquad HP^2 &= (x - a')^2 + (y - b')^2 \end{aligned}\right\}$$

$\therefore$

$$SP^2 - HP^2 = a^2 + b^2 - a'^2 - b'^2 + 2(a' - a)x + 2(b' - b)y$$
$$= \text{also } (SP + HP)(SP - HP)$$
$$= 2A.(2SP - 2A);$$

$\therefore$

$$SP = \frac{a^2 + b^2 - a'^2 - b'^2 + 4A^2}{4A} + \frac{a' - a}{2A}x + \frac{b' - b}{2A}y;$$

$$\therefore \quad SP^2 = \left(\frac{a^2 + b^2 - a'^2 - b'^2 + 4A^2}{4A}\right)^2$$
$$+ \frac{(a' - a)^2}{4A^2}x^2 + \frac{(b' - b)^2}{4A^2}y^2$$
$$+ \frac{a^2 + b^2 - a'^2 - b'^2 + 4A^2}{4A^2}\{(a' - a)x + (b' - b)y\}$$
$$+ \frac{(a' - a)\,b' - b)}{2A^2}xy$$
$$= \text{also } x^2 + y^2 + a^2 + b^2 - 2ax - 2by;$$

$\therefore$ arranging the terms according to the degrees of x and y we have

$$0 = \left\{1 - \left(\frac{a' - a}{2A}\right)^2\right\}x^2 + \left\{1 - \left(\frac{b' - b}{2A}\right)^2\right\}y^2$$
$$- \frac{(a' - a)(b' - b)}{2A^2}xy$$
$$- \left\{2a + (a' - a)\,.\,\frac{a^2 + b^2 - a'^2 - b'^2 + 4A^2}{4A^2}\right\}x$$
$$- \left\{2b + (b' - b)\frac{a^2 + b^2 - a'^2 - b'^2 + 4A^2}{4A^2}\right\}y$$
$$+ a^2 + b^2 - \left(\frac{a^2 + b^2 - a'^2 - b'^2 + 4A^2}{4A}\right)^2,$$

which is the equation required.

126. Cor. 1.—Let the axis of x be S H, and the origin of co-ordinates in C the bisection of S H; then

$$b = o, b' = o \text{ and } a' = -a$$

and the equation becomes

$$\left(1 - \frac{a^2}{A^2}\right)x^2 + y^2 + a^2 - A^2 = 0,$$

or if $\quad A^2 - a^2 = B^2$; then

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1,$$

the same as found directly in p. (95).

127. Cor. 2.—Let the axis of x be S H, and the origin of co-ordinates in the curve at A; then

$$b = o, b' = o, a = SA = AC - CS$$
$$= A - E,$$

and $\quad a' = AH = AC + SC = A + E$;

$\therefore$ the equation becomes

$$\left(1 - \frac{E^2}{A^2}\right)x^2 + y^2$$
$$-\left\{2(A - E) + E\frac{(A-E)^2 - (A+E)^2 + 4A^2}{2A^2}\right\}x = 0,$$

or $(A^2 - E^2)\dfrac{x^2}{A^2} + y^2 = \dfrac{4A^3 - 4A^2E + 4A^2E - 4AE^2}{2A^2}x$

$$= 2\frac{(A^2 - E^2)}{A}x;$$

$$\therefore \qquad \frac{x^2}{A^2} + \frac{y^2}{B^2} = 2 \cdot \frac{x}{A},$$

the same as found directly in p. (96).

In a similar manner the general equation may be reduced to the form of that in which the origin is at the focus S and then axis of x the right line joining the foci.

128. Prop.—*The polar equation of an ellipse, whose foci are* (o, o) *and* $(2\,E, \alpha)$, $2\,E$ *being the right line joining the foci and* α *the traced angle of the vertex, is*

$$r = \frac{a\,(1 - e^2)}{1 + e \cos.\,(\theta - \alpha)},$$

in which a *is the semi-axis major, and* $e = \frac{E}{a}$. (Fig. 31.)

Let A be the vertex, S the focus (o, o) and H the focus $(2\,E, \alpha)$, S being the origin of traced angles; then P being any point (r, θ) of the curve, we have

$$\cos.\,(\theta - \alpha) = \cos.\,A\,S\,P = -\cos.\,P\,S\,H$$

$$= -\frac{S\,P^2 + S\,H^2 - P\,H^2}{2\,S\,P\,.\,S\,H} = -\frac{r^2 + 4E^2 - (2a - r)^2}{2\,r\,.\,2\,E}$$

$$= -\frac{4\,E^2 - 4\,a^2 + 4\,a\,r}{4\,r\,E}$$

$$= \frac{a^2 - E^2 - a\,r}{r\,E}.$$

But $$E = a\,e;$$

$$\therefore\ \cos.\,(\theta - \alpha) = \frac{a^2 - a^2 e^2 - a\,r}{a\,e\,r} = \frac{a\,(1 - e^2) - r}{e\,r};$$

$$\therefore\qquad r = \frac{a\,(1 - e^2)}{1 + e \cos.\,(\theta - \alpha)},$$

the equation required.

129. Cor.—If the origin of traced angles be S H produced; then

$$r = \frac{a\,(1 - e^2)}{1 + e \cos.\,\theta}.$$

130. Def.—*The* Eccentricity *of an ellipse is the ratio of the distance between either focus and centre to that between either vertex and centre.*

Hence eccentricity $= \frac{\text{E}}{a} = e$; which is the reason for the ratio $\frac{\text{E}}{a}$ being called e.

OTHERWISE.

131.—The rectangular equation, when the origin is at the centre and S H the axis of x, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

But

$$x = \text{CM} = \text{SM} - \text{SC} = r \cos. \text{PSM} - \text{E}$$
$$= -r\cos.(\theta - \alpha) - \text{E},$$

also $y = \text{PM} = r \sin.(\theta - \alpha)$,

and from the equation

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$
$$= \frac{b^2}{a^2}(a - x)(a + x);$$

$$\therefore\ r^2 \sin.^2(\theta - \alpha) = \frac{b^2}{a^2}\{a + \text{E} + r\cos.(\theta - \alpha)\} \times$$
$$\{a - \text{E} - r\cos.(\theta - \alpha)\}$$
$$= \frac{b^2}{a^2}\left\{\begin{array}{l}(a^2 - \text{E}^2) + (a - \text{E})\, r\cos.(\theta - \alpha) \\ -(a + \text{E})\, r\cos.(\theta - \alpha) - r^2\cos.^2(\theta - \alpha)\end{array}\right\};$$

$$\therefore\ r^2\{a^2\sin.^2(\theta - \alpha) + b^2\cos.^2(\theta - \alpha)\}$$
$$- 2b^2\text{E}\, r\cos.(\theta - \alpha) = b^2(a^2 - \text{E}^2);$$

$$\therefore\ r^2 + \frac{2b^2\text{E}\cos.(\theta - \alpha)}{a^2\sin.^2(\theta - \alpha) + b^2\cos.^2(\theta - \alpha)}\, r$$
$$= \frac{b^4}{a^2\sin.^2(\theta - \alpha) + b^2\cos.^2(\theta - \alpha)},$$

and solving this quadratic, we have

$$r = \frac{-b^2 E \cos.(\theta-\alpha) \pm b^2\sqrt{\{E^2\cos^2(\theta-\alpha)+a^2\sin^2(\theta-\alpha) + b^2\cos^2(\theta-\alpha)\}}}{a^2\sin.^2(\theta-\alpha)+b^2\cos.^2(\theta-a)}$$

$$= \frac{-b^2 E\cos.(\theta-\alpha)+b^2 a}{a^2-(a^2-b^2)\cos.^2(\theta-\alpha)}$$

$$= b^2 . \frac{-ae\cos.(\theta-\alpha)\pm a}{a^2-a^2e^2\cos.^2(\theta-\alpha)},$$

$$= \frac{b^2}{a} . \frac{1-e\cos.(\theta-\alpha)}{1-e^2\cos.^2(\theta-\alpha)} = \frac{b^2}{a}\frac{1}{1+e\cos.(\theta-\alpha)}$$

$$= \frac{a(1-e^2)}{1+e\cos.(\theta-\alpha)},$$

the labour of which process shows the superiority of the previous direct method.

132. Prop.—*The polar equation of an ellipse whose foci are the points* (E, α) , (E, $\pi+\alpha$) *is*

$$r^2 = \frac{a^2(1-e^2)}{1-e^2\cos.^2(\theta-\alpha)},$$

a *being the semi-axis major, and* e *the eccentricity.* (Fig. 32.)

For, if S and H be the foci and C the centre of the ellipse, then C is the pole, and if S C X$=\alpha$; then C X is the origin of traced angles. Let P be any point (r, θ) of the curve; then

$$SP^2 = SC^2 + CP^2 - 2\,SC\,.\,CP\,.\cos.SCP$$
$$= E^2 + r^2 - 2E\,.\,r\cos.(\theta-\alpha).$$

Also, $HP^2 = HC^2 + CP^2 - 2HC\,.\,CP\,.\cos.HCP$

$$= E^2 + r^2 + 2E\,.\,r\cos.(\theta-\alpha);$$

$$\therefore SP^2 - HP^2 = -4Er\cos(\theta-\alpha) = (SP+HP)(SP-HP).$$

But $SP + HP = 2a$

$\therefore \quad SP - HP = 2SP - 2a;$

$\therefore \; - E r \cos.(\theta - \alpha) = a(SP - a);$

$\therefore \; SP = a - \frac{E}{a} r \cos.(\theta - \alpha);$

and $SP^2 = a^2 - 2 E r \cos.(\theta - \alpha) + \frac{E^2}{a^2} r^2 \cos.^2(\theta - \alpha)$

also, $SP^2 = a^2 + r^2 - 2 a r \cos.(\theta - \alpha)$,

$$\therefore \; r^2 \left\{1 - \frac{E^2}{a^2} \cos.^2(\theta - \alpha)\right\} = a^2 - E^2 .$$

But $\quad E = a e$,

$$\therefore \; r^2 = \frac{a^2(1 - e^2)}{1 - e^2 \cos.^2(\theta - \alpha)},$$

which is the equation required.

OTHERWISE.

The rectangular equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the origin being at the centre and the axis of x the focal axis of the ellipse or axis major.

But $x = CM = - r \cos.(\theta - \alpha)$

and $y = r \sin.\theta$

$$\therefore \; \frac{r^2 \cos.^2(\theta - \alpha)}{a^2} + \frac{r^2 \sin.^2(\theta - \alpha)}{b^2} = 1$$

$$\therefore \; r^2 \{b^2 \cos.^2(\theta - \alpha) + a^2 \sin.^2(\theta - \alpha)\} = a^2 b^2$$

$$\therefore \; r^2 = \frac{a^2 b^2}{b^2 \cos.^2(\theta - \alpha) + a^2 \sin^2(\theta - \alpha)}$$

$$\therefore \quad = \frac{a^2 b^2}{a^2 - (a^2 - b^2) \cos.^2(\theta - \alpha)}$$

$$\therefore \quad = \frac{a^2(a^2 - a^2 e^2)}{a^2 - a^2 e^2 \cos.^2(\theta - \alpha)};$$

$$\therefore \quad = \frac{a^2(1 - e^2)}{1 - e^2 \cos.^2(\theta - \alpha)},$$

as before; which second method is, in this instance, the shorter and preferable.

133. Prop.—*To find the general polar equation of an ellipse when the foci are any points* (R, α), (R', α') *whatever.* (Fig. 33.)

Let S, H be the foci; O the origin of the radius vectors, and O X that of the traced angle;

then $OS = R$, $OH = R'$,

$\angle SOX = \alpha$, $\angle HOX = \alpha'$

and if P be any point in the curve $OP = r$ and $\angle XOP = \theta$.

$$\text{Now,} \quad SP^2 = OS^2 + OP^2 - 2OS.OP.\cos.SOP$$
$$= R^2 + r^2 - 2Rr\cos.(\theta - \alpha),$$
$$HP^2 = OH^2 + OP^2 - 2OH.OP.\cos.HOP$$
$$= R'^2 + r^2 - 2R'r\cos.(\theta - \alpha');$$
$$\therefore SP^2 - HP^2 = R^2 - R'^2 - 2\{R\cos.(\theta - \alpha) - R'\cos.(\theta - \alpha')\}r$$

$$\text{also,} = (SP + HP)(SP - HP) = 2a(2SP - 2a);$$

$$\therefore SP = a + \frac{R^2 - R'^2}{4a} - \frac{R\cos.(\theta - \alpha) - R'\cos.(\theta - \alpha')}{2a}.r;$$

$$\therefore SP^2 = \left(a + \frac{R^2 - R'^2}{4a}\right)^2 - \left(a + \frac{R^2 - R'^2}{4a}\right) \times \frac{R\cos.(\theta - \alpha) - R'\cos.(\theta - \alpha')}{a}r$$

$$+ \frac{\{R \cos.(\theta - \alpha) - R' \cos.(\theta - \alpha')\}^2}{4a^2} r^2$$

also $= R^2 + r^2 - 2 R r \cos.(\theta - \alpha)$

$$\therefore \left\{1 - \frac{\{R \cos.(\theta - \alpha) - R' \cos.(\theta - \alpha')\}^2}{4a^2}\right\} r^2$$

$$- \left\{2 R \cos.(\theta - \alpha) - \left(a + \frac{R^2 - R'^2}{4a}\right) \times \frac{R \cos.(\theta - \alpha) - R' \cos.(\theta - \alpha')}{a}\right\} r$$

$$+ R^2 - \left(a + \frac{R^2 - R'^2}{4a}\right)^2 = 0,$$

the equation required.

134. Cor. 1.—When $R = R'$ and $\alpha' = \pi + \alpha$; the equation becomes

$$\left\{1 - \frac{R^2}{a^2} \cos.^2(\theta - \alpha)\right\} r^2 + R^2 - a^2 = o$$

But in this case $R = a e$, and

$$\therefore r^2 = \frac{a^2(1 - e^2)}{1 - e^2 \cos.^2(\theta - \alpha)},$$

the same as in p. 104.

135. Cor. 2.—When $R = o$, $R' = 2E$ and $\alpha' = \pi + \alpha$, the equation becomes

$$\left\{1 - \frac{E^2}{a^2} \cos.^2(\theta - \pi - \alpha)\right\} r^2 - 2 \frac{(a^2 - E^2) E}{a^2} \times$$

$$\cos.(\pi + \alpha - \theta) . r - \left(\frac{a^2 - E^2}{a}\right)^2 = o;$$

or, since $E = a e$, $\therefore$

$$\{1 - e^2 \cos.^2(\theta - \alpha)\} r^2 + 2(1 - e^2) a e \cos.(\theta - \alpha) . r$$
$$- a(1 - e^2)^2 = 0,$$

$$\therefore r^2 + \frac{2ae(1-e^2)\cos.(\theta-\alpha)}{1-e^2\cos.^2(\theta-\alpha)}r$$

$$= \frac{a^2(1-e^2)^2}{1-e^2\cos.^2(\theta-\alpha)};$$

$\therefore$ Solving the quadratic

$$r = \frac{-ae(1-e^2)\cos.(\theta-\alpha) \pm \sqrt{\{a^2e^2(1-e^2)^2\cos.^2(\theta-\alpha) + a^2(1-e^2)^2(1-e^2)\cos.^2(\theta-\alpha)\}}}{1-e^2\cos.^2(\theta-\alpha)}$$

$$= \frac{-ae(1-e^2)\cos.(\theta-\alpha) \pm a(1-e^2)\sqrt{{}^2\{e^2\cos^2(\theta-\alpha) + 1 - e\cos.^2(\theta-\alpha)\}}}{1-e^2\cos^2(\theta-\alpha)}$$

$$= \frac{a(1-e^2)\{1 \mp e\cos.(\theta-\alpha)\}}{1-e^2\cos.^2(\theta-\alpha)}$$

$$= \frac{a(1-e^2)}{1+e\cos.(\theta-\alpha)},$$

as in art. (131).

Recapitulation of the Equations of an Ellipse.

Rectangular Equations.

136.—1. When the foci are both in the axis of x, and the origin of co-ordinates the bisection of the right line joining them, the rectangular equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

in which a and b are the semi-axes.

2. When the foci are both in the axis of x, and the origin of co-ordinates in the curve, the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2x}{a},$$

a and b being the same as before.

3. When the foci are both in the axis of x and the origin of co-ordinates one of the foci, the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2\sqrt{(a^2 - b^2)}}{a^2}\,x + \frac{b^2}{a^2}$$

a and b being the same as before.

4. Generally, when the foci are the points (a, b) (a', b') and 2 A the constant sum of the distances of any point from the foci, the equation is

$$\left\{1 - \left(\frac{a - a'}{2\,\mathrm{A}}\right)^2\right\}x^2 + \left\{1 - \left(\frac{b - b'}{2\,\mathrm{A}}\right)^2\right\}y^2$$
$$- \frac{(a - a')\,(b - b')}{2\,\mathrm{A}^2}\,xy,$$
$$+ \left\{\frac{a^2 - a'^2 + b^2 - b'^2 + 4\,\mathrm{A}^2}{4\,\mathrm{A}^2}\,(a - a') - 2\,a\right\}x$$
$$+ \left\{\frac{a^2 - a'^2 + b^2 - b'^2 + 4\,\mathrm{A}^2}{4\,\mathrm{A}^2}\,(b - b') - 2\,b\right\}y$$
$$+ a^2 + a'^2 - \left(\frac{a^2 - a'^2 + b^2 - b'^2 + 4\,\mathrm{A}^2}{4\,\mathrm{A}}\right)^2 = 0.$$

Polar Equations.

5. When the foci are the points (o, α), $(2\,\mathrm{E}, \pi + \alpha)$, the polar equation is

$$r = \frac{a^2\,(1 - e^2)}{1 + e^2 \cos\,(\theta - \alpha)},$$

2 a being the axis major, and e the eccentricity and 2E the distance between the foci.

6. When the foci are the points (E, α), $(\mathrm{E}, \pi + \alpha)$, the polar equation is

$$r^2 = \frac{a^2\,(1 - e^2)}{1 - e^2 \cos.^2\,(\theta - \alpha)},$$

a and e being as before,

7. Generally, when (R, α), (R', α'), are the foci and $2E$ the constant sum of the distances of any point of the curve from the foci; the polar equation is

$$\left\{1 - \frac{\{R\cos.(\theta-\alpha) - R'\cos.(\theta-\alpha')\}^2}{4a^2}\right\}r^2$$
$$- \left\{2R\cos.(\theta-\alpha) - \left(a + \frac{R^2 - R'^2}{4a}\right) \times \frac{R\cos.(\theta-\alpha) - R'(\theta-\alpha')}{a}\right\}r$$
$$+ R^2 - \left(a + \frac{R^2 - R'^2}{4a}\right)^2 = 0.$$

Problems on Right Lines, Circles, Parabolas and Ellipses.

137. Prob.—*To find the common points of an ellipse and the co-ordinate axes.*

1. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

be the equation of the ellipse; then the common points with the axes of x and y are respectively

$$(a, o) \text{ and } (o, b).$$

2. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2\frac{x}{a}$; then, making $y = o$, we have

$$x^2 = 2ax;$$
$$\therefore\ x = o \text{ and } 2a.$$

Again, making $x = o$ we have $y = o$,

$\therefore$ the common points with the axes of x, y are

$$(o, o)\ ,\ (2a, o)$$

and similarly for other equations of the ellipse.

138. Prob.—*To find the common points of an ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *and a right line* $\frac{x}{a'} + \frac{y}{b'} = 1$.

At the common points the equations are simultaneous,

$$\left.\begin{array}{r} \therefore\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ \text{and}\ \frac{x}{a'} + \frac{y}{b'} = 1 \end{array}\right\},$$

which simultaneous equations being solved will give two pairs of values of x and y, and these two pairs of values will give two common points.

139. Prob.—*To find the common points of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *and the circle* $(x - a')^2 + (y - b')^2 = R^2$.

At the common points the equations are simultaneous,

$$\left.\begin{array}{r} \therefore\ (x-a')^2 + (y-b')^2 = R^2 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \end{array}\right\},$$

which equations being solved will give four pairs of values of x and y, and $\therefore$ as many common points. The solution leads to that of a Biquadratic.

140. Prob.—*To find the common points of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *and the parabola* $\left(\frac{x}{a'} + \frac{y}{b'}\right)^2 + \frac{x}{c'} + \frac{y}{d'} = 1$.

Solving the simultaneous equations,

$$\left.\begin{aligned} &\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ &\left(\frac{x}{a'} + \frac{y}{b'}\right)^2 + \frac{x}{c'} + \frac{y}{d'} = 1 \end{aligned}\right\}$$

we shall get a biquadratic, which will give four pairs of values of x and y, and $\therefore$ four points.

141. Prob.—*To find the common points of two ellipses having the same centre and the same axis of* x; *that is of the ellipses*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1.$$

Solving

$$\left.\begin{aligned} &\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ &\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1 \end{aligned}\right\}$$

We get

$$\left.\begin{aligned} x &= \pm a\, a' \sqrt{\frac{b^2 - b'^2}{a'^2 b^2 - a^2 b'^2}} \\ y &= \pm b\, b' \sqrt{\frac{a^2 - a'^2}{a^2 b'^2 - a'^2 b^2}} \end{aligned}\right\} \text{which give four common points.}$$

142. Prob.—*Given three radius-vectors* R, R′, R″ *of an ellipse and the* $\angle^s$ (R, R′), (R, R″) *and consequently* (R′, R″) *to find the polar equation of the ellipse.*

Assume the equation to be

$$r = \frac{a(1 - e^2)}{1 + e \cos. \theta};$$

Let $a(1 - e^2) = l$, $\angle$ (R, R′) $= \alpha$, $\angle$ (R, R″) $= \alpha'$; then by the question

$$\left.\begin{aligned} R &= \frac{l}{1 + e\cos.\theta} \\ R &= \frac{l}{1 + e\cos.(\theta + \alpha)} \\ R'' &= \frac{l}{1 + e\cos.(\theta + \alpha')} \end{aligned}\right\};$$

$$\therefore \left.\begin{aligned} \frac{l}{R} - 1 &= e\cos.\theta \\ \frac{l}{R'} - 1 &= e\cos.(\theta + \alpha) \\ \frac{l}{R''} - 1 &= e\cos.(\theta + \alpha') \end{aligned}\right\}.$$

From the first and second equation, and from the first and third, we get

$$\left.\begin{aligned} \frac{R}{R'} \cdot \frac{l - R'}{l - R} &= \frac{\cos.(\theta+\alpha)}{\cos.\theta} = \frac{\cos.\theta.\cos.\alpha - \sin.\theta.\sin.\alpha}{\cos.\theta} \\ &= \cos.\alpha - \sin.\alpha.\tan.\theta \\ \frac{R}{R''} \cdot \frac{l - R''}{l - R} &= \cos.\alpha' - \sin.\alpha'\tan.\theta. \end{aligned}\right\}$$

From these eliminating $\tan.\theta$, we get

$$\frac{R}{l - R}\left(\frac{l - R'}{R'}\sin.\alpha' - \frac{l - R''}{R''}\sin.\alpha\right)$$

$$= \sin.\alpha'.\cos.\alpha - \cos.\alpha'\sin.\alpha = \sin.(\alpha' - \alpha).$$

Hence

$$l\left\{\frac{\sin.(\alpha' - \alpha)}{R} - \frac{\sin.\alpha'}{R'} + \frac{\sin.\alpha}{R''}\right\}$$

$$= \sin.(\alpha' - \alpha) + \sin.\alpha - \sin.\alpha'$$

$$= 2\sin.\tfrac{1}{2}\alpha'\cos.(\tfrac{1}{2}\alpha' - \alpha) - 2\sin.\tfrac{1}{2}\alpha'\cos.\tfrac{1}{2}\alpha' - \alpha$$

$$= 2 \sin. \frac{\alpha'}{2} \left\{ \cos. \left(\frac{\alpha'}{2} - \alpha \right) - \cos. \frac{\alpha'}{2} \right\}$$

$$= 4 \sin. \frac{\alpha'}{2} . \sin. \frac{\alpha' - \alpha}{2} \sin. \frac{\alpha}{2};$$

$$\therefore l = \frac{4 R R' R'' . \sin. \frac{\alpha}{2} \sin. \frac{\alpha'}{2} \sin. \frac{\alpha' - \alpha}{2}}{R R' \sin. \alpha - R R'' \sin. \alpha' + R' R'' \sin. (\alpha - \alpha')}:$$

that is $a\,(1 - e^2)$

$$= \frac{4 R R' R'' \sin. \frac{(R, R')}{2} . \sin. \frac{(R, R'')}{2} \sin. \frac{(R', R'')}{2}}{R R' . \sin. (R, R') - R R'' . \sin (R, R'') + R' R'' . \sin. (R', R'')} . \quad (1.)$$

and it only remains to determine e in order to obtain the constants of the equation required. Now

$$e^2 = \left(\frac{l - R}{R} \right)^2 . \frac{1}{\cos.^2 \theta} = \left(\frac{l - R}{R} \right)^2 . (1 + \tan.^2 \theta),$$

$$\text{and } \tan. \theta = \frac{\cos. \alpha}{\sin. \alpha} - \frac{R}{R'} . \frac{l - R'}{l - R} . \frac{1}{\sin. \alpha}$$

$$= \cot. a - \frac{R}{R'} . \frac{l - R'}{l - R} . \operatorname{cosec}. \alpha;$$

$$\therefore e^2 = \left(\frac{l - R}{R} \right)^2 \left\{ 1 + \cot.^2 \alpha - 2 \cos. \alpha . \operatorname{cosec}.^2 \alpha . \frac{R}{R'} . \frac{l - R'}{l - R} + \frac{R^2}{R'^2} . \left(\frac{l - R'}{l - R} \right)^2 \operatorname{cosec}.^2 \alpha \right\}$$

$$= \left(\frac{l - R}{R} \right)^2 \operatorname{cosec}.^2 \alpha \left\{ 1 - 2 \frac{R}{R'} . \frac{l - R'}{l - R} \cos. \alpha + \frac{R^2}{R'^2} \left(\frac{l - R'}{l - R} \right)^2 \right\};$$

$$\therefore\ e = \frac{l - R}{R} \operatorname{cosec.} \alpha \cdot \sqrt{\left\{1 - 2\frac{R}{R'} \cdot \frac{l - R'}{l - R} \cos. \alpha + \frac{R^2}{R'^2} \cdot \left(\frac{l - R'}{l - R}\right)^2\right\}};$$

$\therefore\ e$ is known and $a\,(1 - e^2)$ has been found ;

$$\therefore\ r = \frac{a\,(1 - e^2)}{1 + e \cos.\, \theta},$$

the required equation is fully determined. Also, a can be found from the values of $a\,(1 - e)^2$ and e.

This is a problem in the planetary theory. For if by observations and theory we find three distances of a planet from the sun (which planet moves in an ellipse about the sun in the focus) and also the angles between those distances, by this process we should determine the equation of the planet's entire orbit.

143. Prob.—*To find the equation of the* Locus *of the extremity of the right line drawn from any point of an ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *parallel to the focal axis, and in a given ratio* (n) *to the distance of that point from the focus.* (Fig. 34.)

Let S, C be the focus and centre of the ellipse, C being also the origin of co-ordinates, and P any point (x, y) of the ellipse. Also, P P′ being drawn parallel to S C, take

$$\frac{S\,P}{P\,P'} = n, \text{ the given ratio.}$$

Then P′ is a point in the required curve or locus.

Let $P'M' = y'$, $C\,M' = -x'$.

Now $y'^2 = P\,M^2 = S\,P^2 - SM^2$

$$= n^2 . \mathrm{P P'^2} - \mathrm{S M^2} = n^2 (x' - x)^2 - (\mathrm{E} - x)^2,$$

E being $=$ S C.

Also, $\frac{x^2}{a^2} + \frac{y'^2}{b^2} = 1$;

$$\therefore \; x = a \sqrt{\left(1 - \frac{y'^2}{b^2}\right)};$$

$\therefore$ by substitution

$$y'^2 = n^2 \left\{x' - a \sqrt{\left(1 - \frac{y'^2}{b^2}\right)}\right\}^2$$

$$- \left\{\mathrm{E} - a \sqrt{\left(1 - \frac{y'^2}{b^2}\right)}\right\}^2;$$

$$\therefore \; n \left\{x' - a \sqrt{\left(1 - \frac{y'^2}{b^2}\right)}\right\}$$

$$= \pm \sqrt{\left[y'^2 + \left\{\mathrm{E} - a \sqrt{\left(1 - \frac{y'^2}{b^2}\right)}\right\}^2\right]}$$

$$= \pm \sqrt{\left\{y'^2 + \mathrm{E}^2 - 2 a \mathrm{E} \sqrt{\left(1 - \frac{y'^2}{b^2}\right)} + a^2 - \frac{a^2}{b^2} y'^2\right\}}$$

$$= \pm \sqrt{\left\{\mathrm{E}^2 + a^2 - \frac{\mathrm{E}^2}{b^2} y'^2 - 2 a \mathrm{E} \sqrt{\left(1 - \frac{y'^2}{b^2}\right)}\right\}}$$

$$= \pm \sqrt{\frac{\mathrm{E}^2 + a^2 - \frac{\mathrm{E}^2}{b^2} y'^2 + b^2 + \frac{\mathrm{E}^2}{b^2} y'^2}{2}}$$

$$\pm \sqrt{\frac{\mathrm{E}^2 + a^2 - \frac{\mathrm{E}^2}{b^2} y'^2 - b^2 - \frac{\mathrm{E}^2}{b^2} y'^2}{2}}$$

by the rule for the extraction of a quadratic binomial surd.

$$\therefore \; n \left\{x' - a \sqrt{\left(1 - \frac{y'^2}{b^2}\right)}\right\} = \pm a \mp \mathrm{E} \sqrt{\left(1 - \frac{y'^2}{b^2}\right)},$$

$$x' = \pm\frac{a}{n} + \left(a \mp \frac{E}{n}\right)\sqrt{\left(1 - \frac{y'^2}{b^2}\right)}\dots\dots\dots(1)$$

which two equations indicate two loci, one generated by drawing PP′ on the left of P; the other by drawing PP″ on the right of P.

But these equations are easily transformed to

$$\frac{\left(x' \mp \frac{a}{n}\right)^2}{\left(a \mp \frac{E}{n}\right)^2} + \frac{y'^2}{b^2} = 1;$$

that is, $x' \mp \frac{a}{n}$ and y being the variable co-ordinates, they belong to ellipses, whose semi-focal and semi-non-focal axes are respectively

$$a - \frac{E}{n}, b; \text{ and } a + \frac{E}{n}, b,$$

their centres also being the points $\left(\frac{a}{n}, o\right)$ and $\left(-\frac{a}{n}, o\right)$.

144. Cor. 1.—When $n = 1$; then these loci are the ellipses

$$\frac{(x-a)^2}{(a-E)^2} + \frac{y^2}{b^2} = 1, \frac{(x+a)^2}{(a+E)^2} + \frac{y^2}{b^2} = 1$$

whose centres are (a, o), $(-a, o)$ and semi-axes $a - E$, b; $a + E$, b respectively, which are represented by Fig. 35.

145. Cor. 2.—When the given ellipse becomes a circle; that is, when $a = b$ and $n = 1$; the loci are

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{a^2} = 1 \text{ and } \frac{(x+a)^2}{a^2} + \frac{y^2}{a^2} = 1$$

which are circles whose centres are (a, o), $(-a, o)$ and whose radii are each $= a$, that of the given circle.

146. Cor. 3.—If n be such a ratio that $a \mp \frac{E}{n} = 0$,

and $\therefore$ $$n = \pm \frac{E}{a},$$

then the equations become

$$x' = \pm \frac{a^2}{E} = \pm \frac{a^2}{\sqrt{(a^2 - b^2)}} \quad \ldots \ldots (2)$$

the equations of two right lines at right angles to the focal axis, and cutting it at the distance of $\frac{a^2}{\sqrt{(a^2 - b^2)}}$ on both sides of the centre.

These right lines are those which, in geometrical treatises of Conic Sections, are called

DIRECTRIX.

Indeed Newton and many others have defined *a conic section to be that plane curve in which the distance of any point of the curve from a given point in the plane is in a given ratio to its $\perp$ distance from a given right line.* The right line in this definition is either of those above found, viz. either

$$x' = \frac{a^2}{\sqrt{(a^2 - b^2)}} \text{ or } x' = -\frac{a^2}{\sqrt{(a^2 - b^2)}}.$$

Observe; that in this case

n is less than unit, and $\therefore$ S P is $<$ P P$'$,

which accords with the geometrical definition of an ellipse.

147. Prob.—*To find the equation of the tangent at any point (a, b) of the ellipse* $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1.$

Let the equation of the secant of the ellipse passing through (a, b), (a'', b'') be

$$\frac{x}{a'''} + \frac{y}{b'''} = 1;$$

then

$$\left.\begin{aligned} \frac{a}{a'''} + \frac{b}{b'''} &= 1 \\ \frac{a''}{a'''} + \frac{b''}{b'''} &= 1 \end{aligned}\right\} \text{which give } a''' = a - b \, . \frac{a - a''}{b - b''},$$

also

$$\left.\begin{aligned} \frac{a^2}{a'^2} + \frac{b^2}{b'^2} &= 1 \\ \frac{a''^2}{a'^2} + \frac{b''^2}{b'^2} &= 1 \end{aligned}\right\}; \ \therefore \frac{a^2 - a''^2}{a'^2} + \frac{b^2 - b''^2}{b'^2} = 0;$$

$$\therefore \qquad \frac{a - a''}{b - b''} = - \frac{a'^2}{b'^2} . \frac{b + b''}{a + a''};$$

$$\therefore \qquad a''' = a + b \, \frac{a'^2}{b'^2} . \frac{b + b''}{a + a''}.$$

Similarly $\quad b''' = b + a \, . \, \dfrac{b'^2}{a'^2} \, . \, \dfrac{a + a''}{b + b''}.$

Let the secant become a tangent; then $a'' = a, b'' = b$, and

$$a''' = a + \frac{b^2}{a} . \frac{a'^2}{b'^2} = \frac{a^2 \, b'^2 + a'^2 \, b^2}{a \, b'^2} = \frac{a'^2 \, b'^2}{a \, b'^2}$$

$$= \frac{a'^2}{a}.$$

Similarly $\qquad b''' = \dfrac{b'^2}{b},$

and the equation of the tangent required is

$$\frac{x}{\left(\frac{a'^2}{a}\right)} + \frac{y}{\left(\frac{b'^2}{b}\right)} = 1.$$

148. Cor.—(Fig. 36.) Since $\frac{a'^2}{a}, \frac{b'^2}{b}$ are the common ordinates of the tangent with the axes of x and y respectively, if T T′ be the tangent at P the point (a, b); then

$$\text{CT} = \frac{a'^2}{a} = \frac{\text{C A}^2}{\text{C M}};$$

$\therefore$ $\quad$ CT : CA :: CA : CM,

which is a geometrical theorem in conic sections that is useful in Newton's 'Principia.'

Similarly $\text{C T}' = \frac{b'^2}{b} = \frac{\text{C B}^2}{\text{P M}} = \frac{\text{C B}^2}{\text{C N}}$,

or $\quad$ CT′ : CB : CB : CN.

149. Prob.—*To find the equation of the normal at any point (a, b) of an ellipse $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$.*

Let $\frac{x}{a''} + \frac{y}{b''} = 1$ be the equation of the tangent,

and $\frac{x}{a'''} + \frac{y}{b'''} = 1$ that of the normal;

then Ex. 2. p. 40, $a''' = a - b \cdot \frac{b''}{a''}$, $b''' = b - a \cdot \frac{a''}{b''}$;

But (148) $a'' = \frac{a'^2}{a}, b'' = \frac{b'^2}{b}$; $\therefore$ $\frac{a''}{b''} = \frac{b}{a} \cdot \frac{a'^2}{b'^2}$;

$$\therefore \quad a''' = a - a \cdot \frac{b'^2}{a'^2} = a\left(1 - \frac{b'^2}{a'^2}\right)$$

$$b''' = b - b \cdot \frac{a'^2}{b'^2} = b\left(1 - \frac{a'^2}{b'^2}\right);$$

$\therefore$ the equation required is

$$\frac{x}{a\left(1-\frac{b'^2}{a'^2}\right)}+\frac{y}{b\left(1-\frac{a'^2}{b'^2}\right)}=1.$$

150. Prob.—*To find the equation between the perpendicular* p, *drawn from the focus upon the tangent at the point (a, b) of an ellipse $\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$, and the radius vector* r *drawn from the focus to that point.*

Let the focus be (a'', b'') and the tangent

$$\frac{x}{a'''}+\frac{y}{b'''}=1;$$

then (Art. 51) $p=\left(\frac{a''}{a'''}+\frac{b''}{b'''}-1\right)\frac{a'''\,b'''}{\sqrt{(a'''^2+b'''^2)}}$.

But
$$a''=-l,\ b''=0,$$
$$a'''=\frac{a'^2}{a},\ b'''=\frac{b'^2}{b};$$

$$\therefore\quad p^2=\frac{(a\,l+a'^2)^2}{a'^4}\cdot\frac{a'^4\,b'^4}{a^2\,b^2\left(\frac{a'^4}{a^2}+\frac{b'^4}{b^2}\right)}$$
$$=\frac{(a\,l+a'^2)^2\,b'^4}{a'^4\,b^2+a^2\,b'^4}.$$

But $\quad r^2=b^2+(a+\mathrm{E})^2$, and $\frac{a^2}{a'^2}+\frac{b^2}{b'^2}=1;$

$$\therefore\quad b^2=b'^2-\frac{b'^2}{a'^2}\cdot a^2.$$

Hence $r^2=b'^2-\frac{b'^2}{a'^2}a^2+a^2+2\,a\,\mathrm{E}+\mathrm{E}^2$

$$=a'^2+\frac{a^2\,\mathrm{E}^2}{a'^2}+2\,a\,\mathrm{E};$$

$$\therefore \qquad a^2 + \frac{2\,a'^2}{E}\,a = \frac{a'^2}{E^2}\,(r^2 - a'^2)\,;$$

and, solving the quadratic,

$$a = -\frac{a'}{E}\,(a' \mp r)\,.$$

Hence

$$b^2 = b'^2\left\{1 - \frac{(a' \mp r)^2}{E^2}\right\} = \frac{b'^2}{E^2}\{E^2 - (a' \mp r)^2\}\,;$$

$$\therefore \qquad a'^4\,b^2 = \frac{a'^4\,b'^2}{E^2}\,\{E^2 - (a' \mp r)^2\},$$

$$\text{and} \qquad a^2\,b'^4 = \frac{a'^2\,b'^4}{E^2}\,(a' \mp r)^2\,;$$

$$\begin{aligned}\therefore\ a'^4\,b^2 + a^2\,b'^4 &= \frac{a'^2\,b'^2}{E^2}\,\{a'^2\,E^2 - (a'^2 - b'^2)\,(\,a' \mp r)^2\}\\ &= a'^2\,b'^2\,(a'^2 - a'^2 \pm 2\,a'\,r - r^2)\\ &= \pm\,(2\,a'\,r \mp r^2)\,.\end{aligned}$$

Also $\quad a\,l + a'^2 = -\,a'^2 \pm a'\,r + a'^2 = \pm\,a'\,r\,;$

$$\therefore \qquad p^2 = \frac{a'^2\,r^2\,.\,b'^4}{a'^2\,.\,b'^2\,(2\,a'\,r \mp r^2)} = \frac{b'^2\,r}{2\,a' \mp r}\,;$$

the upper sign of which is alone applicable to the ellipse; for making $r = a' + E$; then $p = a' + E$, and taking the lower sign, we have

$$a' + E = \frac{b'^2}{3\,a' + E} = \frac{a'^2 - E^2}{3\,a' + E}\,;$$

$$\begin{aligned}\therefore \qquad 3\,a' + E &= a' - E,\\ 4\,a' + 2\,E &= 0,\end{aligned}$$

which is absurd.

$\therefore$ the equation required is

$$p^2 = \frac{b'^2\,r}{2\,a' - r}\,.$$

This equation, like the corresponding one for the parabola, is used by Newton and many other writers in natural philosophy.

151. PROB.—*To find the equation between the perpendicular* p, *drawn from the centre upon the tangent at the point* (a, b) *of an ellipse* $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$, *and the radius vector* r *drawn from the centre to that point.*

As before

$$p = \left(\frac{a''}{a'''} + \frac{b''}{b'''} - 1\right)\frac{a''' b'''}{\sqrt{(a'''^2 + b'''^2)}};$$

whence $a'' = 0,\ b'' = 0,\ a''' = \frac{a'^2}{a},\ b''' = \frac{b'^2}{b}$;

$$\therefore \quad p^2 = \frac{a'''^2 b'''^2}{a'''^2 + b'''^2} = \frac{a'^4 b'^4}{a^2 b^2 \left(\frac{a'^4}{a^2} + \frac{b'^4}{b^2}\right)}$$

$$= \frac{a'^4 b'^4}{a'^4 b^2 + a^2 b'^4};$$

$$\left.\begin{array}{l} \text{But } \therefore\ r^2 = a^2 + b^2 \\ \text{and } \quad \dfrac{a^2}{a'^2} + \dfrac{b^2}{b'^2} = 1 \end{array}\right\} . \quad \therefore\ r^2 = a^2 + b'^2 - \frac{b'^2}{a'^2} a^2$$

$$= a^2 \frac{E^2}{a'^2} + b'^2;$$

$$\therefore \quad a^2 = \frac{(r^2 - b'^2)\, a'^2}{E^2}.$$

Hence

$$b^2 = b'^2 - \frac{b'^2}{a'^2} . a^2 = b'^2 \left(1 - \frac{r^2 - b'^2}{E^2}\right) = b'^2 . \frac{a'^2 - r^2}{E^2};$$

$$\therefore \; a'^4 b^2 + a^2 b'^4 = \frac{a'^4 b'^2}{E^2} \cdot (a'^2 - r^2) + \frac{a'^2 b'^4}{E^2} \cdot (r^2 - b'^2)$$

$$= \frac{a'^2 b'^2}{E^2} \{a'^4 - b'^4 - r^2 \cdot (a'^2 - b'^2)\}$$

$$= \frac{a'^2 b'^2}{E^2} \cdot (a'^2 - b'^2)(a'^2 + b'^2 - r^2)$$

$$= a'^2 b'^2 (a'^2 + b'^2 - r^2);$$

$\therefore$ substituting in the value of p^2, we get

$$p^2 = \frac{a'^2 b'^2}{a'^2 + b'^2 - r^2},$$

the equation required.

The remark made in Art. 150 applies also to this Problem.

152. Prob.—*To find the equation of the osculating circle, or circle of curvature, at any point (a, b) of the ellipse* $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1.$

$$\text{Let } (x - A)^2 + (y - B)^2 = \rho^2$$

be the equation of the *secant circle* passing through the points (a, b), $(a_{,}, b_{,})$, $(a_{,,}, b_{,,})$, in which A, B, and ρ are to be determined, on the supposition that the points $(a_{,}, b_{,})$ $(a_{,,}, b_{,,})$ coincide with (a, b).

Since the secant circle passes through (a, b), $(a_{,}, b_{,})$, $(a_{,,}, b_{,,})$ we have

$$\left.\begin{aligned} (a - A)^2 + (b - B)^2 &= \rho^2 \\ (a_{,} - A)^2 + (b_{,} - B)^2 &= \rho^2 \\ (a_{,,} - A)^2 + (b_{,,} - B)^2 &= \rho^2 \end{aligned}\right\}$$

which, by subtraction or eliminating ρ, give

$$\left.\begin{aligned} 2A(a - a_{,}) + 2B(b - b_{,}) &= a^2 - a_{,}^2 + b^2 - b_{,}^2 \\ 2A(a - a_{,,}) + 2B(b - b_{,,}) &= a^2 - a_{,,}^2 + b^2 - b_{,,}^2 \end{aligned}\right\}$$

But also since (a, b), $(a_{\prime}, b_{\prime})$, $(a_{\prime\prime}, b_{\prime\prime})$ are points of the ellipse, we have

$$\left.\begin{aligned} \frac{a^2}{a'^2} + \frac{b^2}{b'^2} &= 1 \\ \frac{a_{\prime}^2}{a'^2} + \frac{b_{\prime}^2}{b'^2} &= 1 \\ \frac{a_{\prime\prime}^2}{a'^2} + \frac{b_{\prime\prime}^2}{b'^2} &= 1 \end{aligned}\right\} ; \therefore \left.\begin{aligned} b^2 - b_{\prime}^2 &= -\frac{b'^2}{a'^2}(a^2 - a_{\prime}^2) \\ b^2 - b_{\prime\prime}^2 &= -\frac{b'^2}{a'^2}(a^2 - a_{\prime\prime}^2) \\ b_{\prime}^2 - b_{\prime\prime}^2 &= -\frac{b'^2}{a'^2}(a_{\prime}^2 - a_{\prime\prime}^2) \end{aligned}\right\} ;$$

$\therefore$ substituting

$$\left.\begin{aligned} 2\,\mathrm{A}(a - a_{\prime}) - 2\,\mathrm{B}\,.\frac{b'^2}{a'^2}(a - a_{\prime})\frac{a + a_{\prime}}{b + b_{\prime}} &= (a^2 - a_{\prime}^2)\frac{\mathrm{E}^2}{a'^2} \\ 2\,\mathrm{A}(a - a_{\prime\prime}) - 2\,\mathrm{B}\,.\frac{b'^2}{a'^2}(a - a_{\prime\prime})\frac{a + a_{\prime\prime}}{b + b_{\prime\prime}} &= (a^2 - a_{\prime\prime}^2)\frac{\mathrm{E}^2}{a'^2} \end{aligned}\right\} ;$$

$$\therefore \quad \left.\begin{aligned} 2\,\mathrm{A} - 2\,\mathrm{B}\,.\frac{b'^2}{a'^2}\,.\,\frac{a + a_{\prime}}{b + b_{\prime}} &= (a + a_{\prime})\frac{\mathrm{E}^2}{a'^2} \\ 2\,\mathrm{A} - 2\,\mathrm{B}\,.\frac{b'^2}{a'^2}\,.\,\frac{a + a_{\prime\prime}}{b + b_{\prime\prime}} &= (a + a_{\prime\prime})\frac{\mathrm{E}^2}{a'^2} \end{aligned}\right\} ;$$

$\therefore$ by subtraction,

$$2\,\mathrm{B}\,b'^2\left(\frac{a + a_{\prime\prime}}{b + b_{\prime\prime}} - \frac{a + a_{\prime}}{b + b_{\prime}}\right) = \mathrm{E}^2(a_{\prime} - a_{\prime\prime});$$

or
$$2\,\mathrm{B}\left(a\,.\frac{b_{\prime} - b_{\prime\prime}}{a_{\prime} - a_{\prime\prime}} - b - \frac{a_{\prime}b_{\prime\prime} - a_{\prime\prime}b_{\prime}}{a_{\prime} - a_{\prime\prime}}\right)$$
$$= \frac{\mathrm{E}^2}{b'^2}(b + b_{\prime})(b + b_{\prime\prime});$$

or
$$2\,\mathrm{B}\left\{(a + a_{\prime\prime})\frac{b_{\prime} - b_{\prime\prime}}{a_{\prime} - a_{\prime\prime}} - b - b_{\prime\prime}\right\}$$
$$= \frac{\mathrm{E}^2}{b'^2}(b + b_{\prime})(b + b_{\prime\prime});$$

$$\text{or } -2\mathrm{B}\left\{(a + a_{\prime\prime})\frac{b_{\prime}^2}{a'^2}\cdot\frac{a_{\prime} + a_{\prime\prime}}{b_{\prime} + b_{\prime\prime}} + b + b_{\prime\prime}\right\}$$

$$= \frac{\mathrm{E}^2}{b'^2}(b + b_{\prime})(b + b_{\prime\prime});$$

$$\therefore \qquad \mathrm{B} = -\tfrac{1}{2}\mathrm{E}^2 . \frac{a'^2}{b'^2}$$

$$\times \frac{(b + b_{\prime})(b + b_{\prime\prime})(b_{\prime} + b_{\prime\prime})}{(a + a_{\prime\prime})(a_{\prime} + a_{\prime\prime})\,b'^2 + (b + b_{\prime\prime})(b_{\prime} + b_{\prime\prime})\,a'^2} \,..(1).$$

$$\text{Hence} \quad \mathrm{A} = \tfrac{1}{2}(a + a_{\prime})\frac{\mathrm{E}^2}{a'^2}$$

$$-\tfrac{1}{2}\frac{\mathrm{E}^2 . (a + a_{\prime})(b + b_{\prime\prime})(b_{\prime} + b_{\prime\prime})}{(a + a_{\prime\prime})(a_{\prime} + a_{\prime\prime})\,b'^2 + (b + b_{\prime\prime})(b_{\prime} + b_{\prime\prime})\,a'^2}$$

$$= \tfrac{1}{2}(a + a_{\prime})\,\mathrm{E}^2 \times$$

$$\left\{\frac{1}{a'^2} - \frac{(b + b_{\prime\prime})(b_{\prime} + b_{\prime\prime})}{(a+a_{\prime\prime})(a_{\prime}+a_{\prime\prime})\,b'^2+(b+b_{\prime\prime})(b_{\prime}+b_{\prime\prime})\,a'^2}\right\}$$

$$= \tfrac{1}{2}\mathrm{E}^2\frac{b'^2}{a'^2} \times$$

$$\frac{(a + a_{\prime})(a + a_{\prime\prime})(a_{\prime} + a_{\prime\prime})}{(a + a_{\prime\prime})(a_{\prime} + a_{\prime\prime})\,b'^2 \times (b + b_{\prime\prime})(b_{\prime} + b_{\prime\prime})\,a'^2} \,..(2).$$

From these values of A and B the radius of the secant circle may be found, and they also give the position of its centre. But to find the centre and radius of the osculating circle, let the points $(a_{\prime}, b_{\prime})$, $(a_{\prime\prime}, b_{\prime\prime})$ coincide with the point (a, b); then the co-ordinates of the required centre are

$$\mathrm{A} = \tfrac{1}{2}\mathrm{E}^2 . \frac{b'^2}{a'^2}\cdot\frac{8\,a^3}{4\,a^2\,b'^2 + 4\,b^2\,a'^2}$$

$$= \mathrm{E}^2 . \frac{b'^2}{a'^2}\cdot\frac{a^3}{a'^2\,b'^2} = \frac{\mathrm{E}^2\,a^3}{a'^4};$$

and $\mathrm{B} = \frac{1}{2}\mathrm{E}^2 \cdot \dfrac{a'^2}{b'^2} \cdot \dfrac{8\,b^3}{4\,a'^2\,b'^2} = \dfrac{\mathrm{E}^2\,b^3}{b'^4}.$

Hence the radius of the osculating circle is found from

$$\rho^2 = (a - \mathrm{A})^2 + (b - \mathrm{B})^2$$
$$= \left(a - \frac{a^3\,\mathrm{E}^2}{a'^4}\right)^2 + \left(b - \frac{b^3\,\mathrm{E}^2}{b'^4}\right)^2;$$

which, in practice, is sufficiently convenient. But since

$$a - \frac{a^3\,\mathrm{E}^2}{a'^4} = \frac{a}{a'^4} \cdot (a'^4 - a^2\,\mathrm{E}^2)$$
$$= \frac{a}{a'^4}(a'^4 - a^2\,a'^2 + a^2\,b'^2)$$
$$= \frac{a}{a'^4\,b'^2}(a'^4\,b'^2 - a^2\,b'^2 \cdot a'^2 + a^2\,b'^4)$$
$$= \frac{a}{a'^4\,b'^2}\{a'^4\,b'^2 - (a'^2\,b'^2 - a'^2\,b^2)\,a'^2 + a^2\,b'^4\}$$
$$= \frac{a}{a'^4\,b'^2}(a^2\,b'^4 + a'^4\,b^2);$$

and similarly $b - \dfrac{b^3\,\mathrm{E}^2}{b'^4} = \dfrac{b}{a'^2\,b'^4}(a^2\,b'^4 + a'^4\,b^2);$

$$\therefore \quad \rho^2 = (a^2\,b'^4 + a'^4\,b^2)^2 \cdot \left(\frac{a^2}{a'^8\,b'^4} + \frac{b^2}{a'^4\,b'^8}\right)$$
$$= \frac{(a^2\,b'^4 + a'^4\,b^2)^3}{a'^8\,b'^8};$$

$$\therefore \quad \rho = \frac{(a^2\,b'^4 + a'^4\,b^2)^{\frac{3}{2}}}{a'^4\,b'^4}.$$

It may also be thus changed to the form given in geometrical systems of Conic Sections:

$$\rho = \frac{1}{a'\,b'} \cdot \left(\frac{a^2\,b'^4 + a'^4\,b^2}{a'^2\,b'^2}\right)^{\frac{3}{2}}$$

$$= \frac{1}{a'\,b'} \cdot \left(\frac{a^2}{a'^2} \cdot b'^2 + \frac{b^2}{b'^2} \cdot a'^2\right)^{\frac{3}{2}}$$

$$= \frac{1}{a'\,b'}\left\{\left(1 - \frac{b^2}{b'^2}\right) b'^2 + \left(1 - \frac{a^2}{a'^2}\right) a'^2\right\}^{\frac{3}{2}}$$

$$= \frac{(a'^2 + b'^2 - a^2 - b^2)^{\frac{3}{2}}}{a'\,b'};$$

which, expressed geometrically, is

$$\rho = \frac{(\mathrm{A\,C}^2 + \mathrm{B\,C}^2 - \mathrm{C\,P}^2)^{\frac{3}{2}}}{\mathrm{A\,C} \times \mathrm{B\,C}}.$$

153. Prob.—*To find the chord of the osculating circle at the point (a, b) of an ellipse $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$, which passes through that point and the centre of the ellipse.* (Fig. 37.)

If PG be the diameter of the circle of curvature at the point P of the ellipse whose centre is C and P V the chord required; then C y being $\perp$ to the tangent at P, the right $\angle$d Δs, CyP, P V G are similar, and

$$\mathrm{P\,V} : \mathrm{PG} :: \mathrm{C}y : \mathrm{C\,P}$$

or $$\mathrm{P\,V} : 2\rho :: p : \sqrt{(a^2 + b^2)};$$

$$\therefore\ \mathrm{P\,V} = \frac{2\,p\,\rho}{\sqrt{(a^2 + b^2)}} = \frac{2}{\sqrt{(a^2 + b^2)}} \times$$

$$\frac{a'\,b'}{\sqrt{(a'^2 + b'^2 - a^2 - b^2)}} \times \frac{(a'^2 + b'^2 - a^2 - b^2)^{\frac{3}{2}}}{a'\,b'}$$

$$= 2 \cdot \frac{a'^2 + b'^2 - a^2 - b^2}{\sqrt{(a^2 + b^2)}};$$

which agrees with the expression in geometrical treatises.

PROB.—*To find that chord of the osculating circle at the point* (a, b) *of an ellipse* $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$ *which passes through* (a, b) *and the focus* $(-E, o)$.

Let S be the focus, $Sy' \perp$ to the tangent and PV′ the chord required; then the $\triangle$s $Sy'P$, PGV' are similar, and

$$PV' : PG :: Sy' : SP$$

$$\therefore PV' = \frac{PG \times Sy'}{SP} = \frac{2(a'^2 + b'^2 - a^2 - b^2)^{\frac{3}{2}}}{a'b' \cdot r} \cdot \sqrt{\frac{b'^2 r}{2a' - r}}$$

$$= 2 \cdot \frac{(a'^2 + b'^2 - a^2 - b^2)^{\frac{3}{2}}}{a' \sqrt{(2a'r - r^2)}}.$$

But $r^2 = b^2 + (a + E)^2$

$$= a^2 + b^2 + 2aE + E^2$$

$$= a^2 + b'^2 - \frac{b'^2}{a'^2} a^2 + 2aE + a'^2 - b'^2$$

$$= \frac{a^2 \cdot E^2}{a'^2} + 2aE + a'^2$$

$$= \frac{a'^4 + 2a'^2 aE + a^2 E^2}{a'^2};$$

$$\therefore r = \frac{a'^2 + aE}{a'},$$

and $2a'r - r^2 = \frac{a'^2 + aE}{a'} \cdot \left(2a' - \frac{a'^2 + aE}{a'}\right)$

$$= \frac{a'^4 - a^2 E^2}{a'^2}$$

$$= \frac{a'^4 - a^2 a'^2 + a^2 b'^2}{a'^2}$$

$$= \frac{a'^4 - a^2 a'^2 + a'^2 b'^2 - a'^2 b^2}{a'^2}$$

$$= a'^2 + b'^2 - a^2 - b^2\,;$$

$$\therefore\ \mathrm{PV} = 2\,.\,\frac{a'^2 + b'^2 - a^2 - b^2}{a'}\,;$$

which is the value of the geometric expression.

These several expressions for the Radius and Chords of the Osculating Circle are much used in Mechanics and Astronomy.

154. Prob.—*To find the equation of the Evolute of the ellipse* $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$ *at the point* (a, b).

As in p. (125) from the equations

$$\left.\begin{array}{l} \mathrm{A} = \dfrac{\mathrm{E}^2 a^3}{a'^4},\ \mathrm{B} = \dfrac{\mathrm{E}^2 b^3}{b'^4} \\[2ex] \text{and } \dfrac{a^2}{a'^2} + \dfrac{b^2}{b'^2} = 1 \end{array}\right\}$$

we must eliminate a and b. But

$$a = \frac{\mathrm{A}\,.^{\frac{1}{3}}\,a'^{\frac{4}{3}}}{l^{\frac{2}{3}}},\ b = \frac{\mathrm{B}\,.^{\frac{1}{3}}\,b'^{\frac{4}{3}}}{l^{\frac{2}{3}}};$$

$\therefore$ substituting in the third equation, we get

$$\left(\frac{a'}{l^2}\cdot \mathrm{A}\right)^{\frac{2}{3}} + \left(\frac{b'}{l^2}\cdot \mathrm{B}\right)^{\frac{2}{3}} = 1,$$

or since (A , B) is the point of the evolute corresponding to *any* point (a, b) of the ellipse, A and B are variable and the equation required is

$$\left(\frac{a'}{a'^2 - b'^2}\cdot x\right)^{\frac{2}{3}} + \left(\frac{b'}{a'^2 - b'^2}\cdot y\right) = 1.$$

We shall in this place give the equations of the involute of a circle.

155. PROB.—*To find the equation of the Involute of a given circle $x'^2 + y'^2 = R^2$; that is, of the curve, whose Evolute is this circle.* (Fig. 38.)

Let the point (x, y) or P in the involute A P, correspond to the point (x', y') or P′ in the given circle $x'^2 + y'^2 = R^2$ or evolute; then since P P′ is the radius of osculation at the point P, P′ being the centre of the osculating circle at that point, P P′ must be a tangent to the given circle at P′. Hence the equation to P′ P is

$$\frac{x}{\left(\frac{R^2}{x'}\right)} + \frac{y}{\left(\frac{R^2}{y'}\right)} = 1, \text{ or } x x' + y y' = R^2.$$

$$\text{Also, since } P' P = P' A = C P' . < P' C A$$
$$= R . \cos.^{-1} \frac{x'}{R},$$
$$\text{and } P' P^2 = (x' - x)^2 + (y' - y)^2$$

$$\left.\begin{array}{ll} \therefore\ (x' - x)^2 + (y' - y)^2 = R^2 \left(\cos.^{-1} \frac{x'}{R}\right)^2; & \\ \text{But} \quad x'^2 + y'^2 = R^2 & \\ \text{and} \quad x x' + y y' = R^2, & \end{array}\right\}$$

from which three equations it only remains to eliminate x' and y'. The first becomes by reduction and substitution

$$x^2 + y^2 - R^2 = R^2 \left(\cos.^{-1} \frac{x'}{R}\right)^2,$$

and the other two give

$$\frac{x'^2}{R^2} - \frac{2 R x}{x^2 + y^2} \frac{x'}{R} = \frac{y^2 - R^2}{x^2 + y^2};$$

$$\therefore \quad \frac{x'}{R} = \frac{R x \pm y \sqrt{(a^2 + y^2 - R^2)}}{x^2 + y^2}.$$

Hence the required equation is

$$\sqrt{\frac{\mathrm{R}\,x \pm y\sqrt{(x^2 + y^2 - \mathrm{R}^2)}}{x^2 + y^2}} = \cos.\ \frac{x^2 + y^2 - \mathrm{R}^2}{\mathrm{R}^2}.$$

156. Cor.—Hence the polar equation, the centre of the circle being the pole, is (since $x = r\cos.\theta$, and $y = r\sin.\theta$)

$$\sqrt{\frac{\mathrm{R}\cos.\theta \pm \sin.\theta\sqrt{(r^2 - \mathrm{R}^2)}}{r}} = \cos.\ \frac{r^2 - \mathrm{R}^2}{\mathrm{R}^2}.$$

The equation between the perpendicular p on the tangent from C and the radius vector r is

$$p = \sqrt{(r^2 - \mathrm{R}^2)}.$$

For P P′ being the radius of curvature of the Involute at P, it is $\perp$ to the tangent at P.

$$\therefore\ p = \mathrm{C}y = \mathrm{PP}' = \sqrt{(\mathrm{CP}^2 - \mathrm{CP}'^2)}$$
$$= \sqrt{(r^2 - \mathrm{R}^2)}.$$

This curve is of use in the formation of the teeth of wheels in Mechanics.

157. Prob.—*To find the area of an ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *between the points* $(-a, o)$, (a, o). (Fig. 39.)

Let A P V be the ellipse and A Q V a semi-circle on A V. Draw the rectangles P M′, Q M′ having a common base M M′. Then

$$\square\,\mathrm{PM}' : \square\,\mathrm{QM}' :: \mathrm{PM} : \mathrm{QM}$$
$$:: \frac{b}{a}\sqrt{(a^2 - \mathrm{CM}^2)} : \sqrt{(a^2 - \mathrm{CM}^2)}$$
$$:: b : a,$$

which being the same for all pairs of like rectangles, by making their number infinite, we get

$$\text{area APV} : \text{area AQV} :: b : a;$$

$$\text{but area AQV} = \frac{\pi a^2}{2};$$

$$\therefore \text{ area APV} = \frac{\pi}{2} a b.$$

It will be found in like manner that the area of the ellipse, on the other side of A V, is $\frac{\pi}{2} a b$.

$\therefore$ the whole area of the ellipse is

$$\pi \,.\, ab.$$

The area of the ellipse is of use in many questions, but especially in astronomy, and the same may be said of that of the parabola.

158. Prob.—*To find the sectorial area of an ellipse* $r = \dfrac{a(1-e^2)}{1+e\cos.\theta}$, *which is comprised between the radius vectors* R, R'. (Fig. 40.)

Let PSP' be the area required; $\angle ASP' = \alpha'$, $\angle ASP = \alpha$ and $\angle SCQ = u$; M P being produced to meet the circle upon the same diameter as the ellipse. Then

$$\text{area } PSP' = ASP' - ASP;$$

$$\text{but } ASP = ACP - \triangle SCP = ACQ \,.\, \frac{b}{a} - SCQ \,.\, \frac{b}{a}$$

$$= \frac{b}{2a}(a^2 u - a^2 e \sin.u) = \frac{ab}{2}(u - e\sin.u),$$

an expression useful in astronomy. Hence

$$ASP' = \frac{ab}{2}(u' - e\sin.u');$$

$\therefore$ Sector P S P$' = \frac{ab}{2}\{u' - u - e(\sin u' - \sin. u)\}$.

Again $\sin. u = \frac{QM}{a} = \frac{PM}{b} = \frac{R \sin. \alpha}{b}$

$$= \frac{a(1-e^2)\sin \alpha}{b(1+e\cos. \alpha)}$$

$$= \frac{a(1-e^2)\sin. \alpha}{a\sqrt{(1-e^2)}(1+\cos. \alpha)} = \frac{\sqrt{(1-e^2)}\sin. \alpha}{1+e\cos. \alpha}.$$

Similarly $\sin. u' = \frac{\sqrt{(1-e^2)}\sin. \alpha'}{1+e\cos. a}$.

Also

$$\cos. u = \frac{CM}{a} = \frac{ae - SM}{a} = e + \frac{R}{a}\cos. \alpha$$

$$= \frac{e+\cos. \alpha}{1+e\cos. \alpha},$$

and $\cos. u' = \frac{e+\cos. \alpha'}{1+e\cos. \alpha'}$.

Hence, by substituting these several values, we first get

$\cos. (u'-u)$

$$= \frac{\cos.(\alpha'-\alpha)+e(\cos. \alpha+\cos. \alpha)+e^2(1-\sin. \alpha'.\sin. \alpha)}{(1+e\cos. \alpha')(1+e\cos. \alpha)},$$

$\sin. u' - \sin. u$

$$= \frac{\sin. \alpha' - \sin. \alpha - e(\cos. \alpha' - \cos. \alpha)}{(1+e\cos. \alpha')(1+e\cos. \alpha)},$$

and thence

$$\text{Sector P S P}' = \frac{ab}{2}\left\{\cos.^{-1}\frac{\cos.(\alpha'-\alpha)+e(\cos. \alpha'+\cos. \alpha) + e^2(1-\sin. \alpha'\sin. \alpha)}{(1+e\cos. \alpha')(1+e\cos. \alpha)}\right.$$

$$\left. - e.\frac{\sin. \alpha' - \sin. \alpha - e(\cos. \alpha' - \cos. \alpha)}{(1+e\cos. \alpha')(1+e\cos. \alpha)}\right\},$$

which is expressed in traced angles, α, α', independently of the corresponding radius vectors.

Ex. Let the area between $R = a - a e$, and $R' = a + a e$ be required. Then $\alpha = o$, $\alpha' = \pi$ and the expression becomes

$$\text{Sector} = \frac{ab}{2}\left\{\cos.^{-1}\frac{\cos.\pi + e(\cos.\pi - \cos.o) + e^2(1 - \sin.\pi.\sin.o)}{(1 + e\cos.\pi)(1 + e\cos.o)}\right.$$
$$\left. - e.\frac{\sin.\pi - \sin.o - e(\cos.\pi - \cos.o)}{(1 + e\cos.\pi)(1 + e\cos.o)}\right\},$$

$$= \frac{ab}{2}.\cos.^{-1}\frac{-1 + e^2}{1 - e^2} = \frac{ab}{2}.\cos.^{-1}(-1) = ab.\frac{\pi}{2},$$

which we have already shown to be the correct value.

159. Cor.—Since $2\sin.^2\frac{1}{2}u = 1 - \cos.u$

$$= 1 - \frac{e + \cos.\alpha}{1 + e\cos.\alpha} = \frac{(1 - e)(1 - \cos.\alpha)}{1 + e\cos.\alpha},$$

and $2\cos.^2\frac{1}{2}u = 1 + \cos.u$

$$= 1 + \frac{e + \cos.\alpha}{1 + e\cos.\alpha} = \frac{(1 + e)(1 + \cos.\alpha)}{1 + e\cos.\alpha},$$

$$\therefore\ \tan.^2\tfrac{1}{2}u = \frac{1 - e}{1 + e}.\frac{1 - \cos.\alpha}{1 + \cos.\alpha} = \frac{1 - e}{1 + e}.\tan.^2\tfrac{1}{2}\alpha;$$

$$\therefore\ \tan.\tfrac{1}{2}u = \sqrt{\frac{1 - e}{1 + e}}.\tan.\tfrac{1}{2}\alpha,$$

which is used in astronomy.

160. Prob.—*To extend* Lambert's Theorem *in the parabola to the ellipse; that is, to find the area of the sector comprised by* R, R′ *in terms of* R, R′ *and* c, c *being the chord of the arc between* R, R′.

As before, area S P P$' = \dfrac{ab}{2}\{u'-u-e(\sin. u'-\sin. u)\}$,

$$\cos. (\alpha'-\alpha) = \frac{R^{2\prime} + R^2 - c^2}{2 R R'};$$

$$\therefore \cos.^2 \frac{\alpha'-\alpha}{2} \quad \frac{1 + \cos. (\alpha'-\alpha)}{2}$$

$$= \frac{(R' + R + c)(R' + R - c)}{4 R R'};$$

$$\therefore R' R \cos.^2 \frac{\alpha'-\alpha}{2} = \tfrac{1}{2} \cdot (R'+R+c)\tfrac{1}{2}(R'+R-c).$$

$$\text{Let } \tfrac{1}{2}(R' + R + c) = a(1 - \cos. \beta'),$$
$$\tfrac{1}{2}(R' + R - c) = a(1 - \cos. \beta):$$

then since $R = a(1 - e \cos. u)$, $R' = a(1 - e \cos. u)$

$\therefore R'R = a\{1 - e(\cos. u' + \cos. u) + e^2 \cos. u' . \cos. u\}$,

and $R R' \cos. (\alpha' - \alpha) = R' R \cos. \alpha' \cos. \alpha$

$$+ R' R \sin. \alpha' \sin. \alpha = x'x + y'y$$

$$= a^2 (e - \cos. u)(e - \cos. u') + a^2 (1-e^2) \sin. u' . \sin. u$$
$$= a^2 \{\cos. (u' - u) - e(\cos. u' + \cos. u)$$
$$+ e^2 (1 - \sin. u' . \sin. u)\};$$

$$\therefore R'R . \{1 + \cos. (\alpha' - \alpha)\} = a^2 \{1 + \cos. (u' - u)\}$$
$$- 2e(\cos. u' + \cos. u) + e^2 \{1 + \cos. (u' + u)\};$$

$$\therefore R'R . \cos.^2 \tfrac{1}{2}(\alpha'-\alpha) = a^2 \{\cos.^2 \tfrac{1}{2}(u'-u)$$
$$- 2e \cos. \tfrac{1}{2}(u' + u) \cos. \tfrac{1}{2}(u' - u) + e^2 \cos.^2 \tfrac{1}{2}(u'+u)\};$$

$$\therefore \quad = \text{also } (1 - \cos. \beta')(1 - \cos. \beta)$$
$$= \left(2 \sin. \frac{\beta'}{2} . \sin. \frac{\beta}{2}\right)^2;$$

$$\therefore \cos. \tfrac{1}{2}(u'-u) - e \cos. \tfrac{1}{2}(u'+u) = 2 \sin. \frac{\beta'}{2} \sin. \frac{\beta}{2} \ (1.);$$

also, $\frac{1}{a}(\mathrm{R}' + \mathrm{R}) = 2 - \cos.\beta' - \cos.\beta$

$$= 2 - e(\cos.u' + \cos.u);$$

$$\therefore \cos.\beta' + \cos.\beta = e(\cos.u' + \cos.u);$$

$$\therefore 4e\cos.\tfrac{1}{2}(u' + u)\cos.\tfrac{1}{2}(u' - u) = 2(\cos.\beta' + \cos.\beta).$$

This combined with equation (1) will give

$$\{\cos.\tfrac{1}{2}(u' - u) + e\cos.\tfrac{1}{2}(u' + u)\}^2$$
$$= (2\cos.\tfrac{1}{2}\beta.\cos.\tfrac{1}{2}\beta')^2;$$

$$\therefore \cos.\tfrac{1}{2}(u' - u) + e\cos.\tfrac{1}{2}(u' + u)$$
$$= 2\cos.\tfrac{1}{2}\beta'.\cos.\tfrac{1}{2}\beta.$$

But $\cos.\tfrac{1}{2}(u' - u) - e\cos.\tfrac{1}{2}(u' + u)$

$$= 2\sin.\tfrac{1}{2}\beta'.\sin.\tfrac{1}{2}\beta;$$

$$\therefore u' - u = \beta' - \beta;$$

$$\therefore \sin.(u' - u) = \sin.(\beta' - \beta);$$

$$\therefore e\sin.\tfrac{1}{2}(u' - u)\cos.\tfrac{1}{2}(u + u)$$
$$= \sin.\tfrac{1}{2}(\beta' - \beta)\cos.\tfrac{1}{2}(\beta' + \beta),$$

or $e(\sin.u' - \sin.u) = \sin.\beta' - \sin\beta.$

Consequently

$$\text{area} = \tfrac{1}{2}ab\{\beta' - \beta - (\sin.\beta' - \sin.\beta)\},$$

in which

$$\sin.\tfrac{1}{2}\beta' = \tfrac{1}{2}\left(\frac{\mathrm{R}' + \mathrm{R} + c}{a}\right)^{\frac{1}{2}},$$

and $\sin.\tfrac{1}{2}\beta = \tfrac{1}{2}\left(\frac{\mathrm{R}' + \mathrm{R} - c}{a}\right)^{\frac{1}{2}}.$

This expression is to be found also in works on Astronomy.

SECTION VI.

THEORY OF THE HYPERBOLA IN A CO-ORDINATE PLANE.

161. DEF.—*A* HYPERBOLA *is a curve, every point of which is in one plane, and such that the difference of the distances of that point from two given points in that plane, is constant.* (Fig. 41.)

If S and H be the given or fixed points, and P, P′ any points in the hyperbola; then will

$$\mathrm{H\,P} \sim \mathrm{S\,P} = \mathrm{H\,P'} \sim \mathrm{S\,P'}$$

and so on for any other points; that is, the excess of the distance of P from H above its distance from S is constant.

162. DEF.—The *Foci* of a hyperbola are the two given points S, H.

163. PROP.—*The equation of a hyperbola, when the foci are the points* $(-\mathrm{E}, o)$, (E, o), ($2\,\mathrm{E}$ *being the distance between the foci*), *is*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

in which 2 a *is the constant excess of* H P *above* S P, *and* $b^2 = \mathrm{E}^2 - a^2$. (Fig. 42.)

Let S, H be the foci, and $\mathrm{S\,H} = 2\,\mathrm{E}$.

Also, by the definition,

$$\mathrm{H\,P} - \mathrm{S\,P} = \text{constant} = 2\,a \text{ suppose.}$$

But $\mathrm{H\,P}^2 = \mathrm{H\,M}^2 + \mathrm{P\,M}^2 = (\mathrm{E} + x)^2 + y^2$

$\mathrm{S\,P}^2 = \mathrm{S\,M}^2 + \mathrm{P\,M}^2 = (x - \mathrm{E})^2 + y^2$

$$\therefore\ HP^2 - SP^2 = 4Ex = (HP - SP)(HP + SP)$$
$$= 2A\,.\,(2A + 2SP);$$
$$\therefore\quad SP = \frac{E}{a}x - a;$$
$$\therefore\quad SP^2 = \frac{E^2}{a^2}x^2 - 2Ex + A^2 = \text{also } (x - E)^2 + y^2;$$
$$\therefore\quad x^2\left(1 - \frac{E^2}{a^2}\right) + y^2 = a^2 - E^2;$$
$$\therefore\quad \frac{x^2}{a^2} - \frac{y^2}{E^2 - a^2} = 1;$$
$$\text{or, } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the equation required.

164. Def.—The Vertex of a Hyperbola is the common point of the hyperbola, and the right line joining the foci.

Thus A is the vertex of the hyperbola A P.

165. Def.—The Axis of a Hyperbola is the distance between the vertices.

166. Def.—The Centre of a Hyperbola is the point which bisects the distance between the foci.

167. Prob.—*To trace the figure of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, *and to find the maximum and minimum values of its co-ordinates.* (Fig. 41.)

Let $y = 0$; then $x = \pm a = CA, CA'$ suppose.

Let $\pm x$ be less than $\pm a$; then $-\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$ is positive.

$\therefore$ y is imaginary.

Hence the minimum values of x are a and $-a$.

Again, let $x = +\infty$; then $y = \pm\infty$,

let $x = -\infty$; then $y = \pm\infty$.

Hence the curve consists of two symmetrical infinite branches on the different sides of CX and of two others exactly like those on the different sides of CX′; and by taking any number of successive values of x, and finding the corresponding values of y, any number of points in the curve may be found, and the curve traced to any required degree of practical accuracy.

DEF.—The branches of P′ A′ Q′ constitute what is called the OPPOSITE HYPERBOLA.

168. PROB.—*To trace the hyperbola, whose foci,* S, H *are given, and in which the constant difference* $2a$ *is given, by points.* (Fig. 44.)

First, take A so that

$$HA - SA = 2a = AA'.$$

Draw HL an indefinite and fixed right line in which assume $HD = 2a$, and from centre H draw any number of concentric arcs pq, $p'q'$, &c. Also, from centre S with radii Dp, Dp', Dp'', &c., draw arcs cutting the former arcs respectively in P, P′, P″, P‴, &c.

Then P, P′, P″, &c., are points in the curve; for $HP - SP = Hp - Dp = HD = 2a$ and so on for the other points.

Thus, by means of a pair of compasses a great number of points may be soon found, and the branch of the curve described with considerable accuracy.

169. Prop.—*The equation of a hyperbola whose foci are the points* $(E - a, o)$, $\{-(E + a), o\}$, a *being the axis and* 2 E *the distance between the foci is*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \mp 2 . \frac{x}{a},$$

in which $b^2 = E^2 - a^2$. (Fig. 43.)

Let S and H be the foci and P any point in the curve; then *o* being the origin of co-ordinates and C the bisection of S H, &c.

$$O C = a, \quad S C = E.$$

Also
$$H P^2 = H M^2 + P M^2 = (E + a + x)^2 + y^2$$
$$S P^2 = S M^2 + P M^2 = (x - E + a)^2 + y^2;$$
$$\therefore \; H P^2 - S P^2 = 4(a + x) E = (H P - S P) \times (H P + S P) = 2a . (2a + 2 S P;)$$
$$\therefore \quad S P = E - a + \frac{E}{a} x;$$
$$\therefore \; S P^2 = (E - a)^2 + 2 \frac{E}{a} (E - a) x + \frac{E^2}{a^2} x^2 .$$

Also
$$= x^2 + (E - a)^2 - 2 (E - a) x + y^2;$$
$$\therefore \; \left(\frac{E^2}{a^2} - 1\right) x^2 - y^2 = -2 (E - a) \left(1 + \frac{E}{a}\right) x$$
$$= -2 \frac{b^2}{a} x;$$
$$\therefore \; \frac{b^2}{a^2} x^2 - y^2 = -2 \frac{b^2}{a} x,$$
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -2 . \frac{x}{a}.$$

The plus sign obtains for the opposite hyperbola.

170. Prob.—*The equation of a hyperbola whose foci are*

the points $(o, \dot{o})$, $(2\,E, o)$, $2\,E$ *being the right line joining the foci, is*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{\sqrt{(a^2+b^2)}}{a^2}x - \frac{b^2}{a^2},$$

in which $2\,a$ *is the axis, and* $b^2 = E^2 - a^2$. (Fig. 45.)

For the focus S, in this case, being the origin of co-ordinates, we have

$$H\,P^2 = H\,M^2 + P\,M^2 = (-2\,E + x)^2 + y^2,$$

$$S\,P^2 = S\,M^2 + P\,M^2 = x^2 + y^2;$$

$$\therefore\ HP^2 - SP^2 = 4\,E^2 - 4\,E\,x = (HP - SP)\,(HP + SP)$$
$$= 2\,a\,.\,(2a + 2\,S\,P)\,;$$

$$\therefore\ S\,P = \frac{E^2}{a} - a - \frac{E}{a}x = \frac{b^2}{a} - \frac{E}{a}x;$$

$$\therefore\ S\,P^2 = \frac{b^4}{a^2} - 2\,\frac{b^2\,E}{a^2}x + \frac{E^2}{a^2}x^2$$

$$= \text{also } x^2 + y^2;$$

$$\therefore\ x^2\,.\left(\frac{E^2}{a^2} - 1\right) - y^2 = 2\,.\frac{b^2\,E}{a^2}x - \frac{b^4}{a^2};$$

$$\therefore\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 2\frac{\sqrt{(a^2+b^2)}}{a^2}x - \frac{b^2}{a^2},$$

the equation required.

171. Prop.—*Given the foci* (a, b) (a', b') *of a hyperbola, and the constant difference of the distance of any point* (x, y) *from the points* (a, b) (a', b'), *to find the equation of the hyperbola.* (Fig. 46.)

Let P be the point (x, y) in the hyperbola, and S H the foci (a, b), (a', b'); then

$$H\,P^2 = (x - a')^2 + (y - b')^2,$$

$$S\,P^2 = (x - a)^2 + (y - b)^2;$$

$$\therefore\ \text{H P}^2 - \text{S P}^2 = a'^2 + b'^2 - a^2 - b^2 + 2(a - a')\,x$$
$$+ 2\,(b - b')\,y,$$

also $= (\text{HP} - \text{SP})\,(\text{HP} + \text{SP}) = 2\,\text{A}\,(2\,\text{B} + 2\,\text{SP})$

$$\therefore\ \text{SP} = \frac{a'^2 + b'^2 - a^2 - b^2 - 4\,\text{A}^2}{4\,\text{A}} + \frac{a - a'}{2\,\text{A}}\,x + \frac{b - b'}{2\,\text{A}}\,y.$$

Hence $\quad x^2 + y^2 - 2\,a\,x - 2\,b\,y + a^2 + b^2$

$$= \frac{(a'^2 + b'^2 - a^2 - b^2 - 4\,\text{A}^2)^2}{16\,\text{A}^2} + \frac{(a - a')^2}{4\,\text{A}^2}\,x^2$$

$$+ \frac{(b - b')^2}{4\,\text{A}^2}\,y^2 + \frac{a'^2 + b'^2 - a^2 - b^2 - 4\,\text{A}^2}{4\,\text{A}^2} \times$$

$$\{(a - a')\,x + (b - b')y\} + \frac{(a - a')\,(b - b')}{4\,\text{A}^2}\,x\,y;$$

which being arranged, gives

$$0 = \left\{1 - \left(\frac{a' - a}{2\,\text{A}}\right)^2\right\} a^2 + \left\{1 - \left(\frac{b' - b}{2\,\text{A}}\right)^2\right\} y^2$$

$$- \frac{(a' - a)\,(b' - b)}{4\,\text{A}^2}\,x\,y$$

$$- \left\{2\,a + (a' - a)\,.\,\frac{a^2 + b^2 - a'^2 - b'^2 + 4\,\text{A}^2}{4\,\text{A}^2}\right\} x$$

$$- \left\{2\,b + (b' - b)\,.\,\frac{a^2 + b^2 - a'^2 - b'^2 + 4\text{A}^2}{4\,\text{A}^2}\right\} y$$

$$+\ a^2 + b^2 - \left(\frac{a^2 + b^2 - a'^2 - b'^2 + 4\,\text{A}^2}{4\,\text{A}}\right)^2,$$

the equation required, which is precisely of the same form as the general equation of the ellipse.

172. Cor. 1.—Let the axis of x be S H, and the origin of co-ordinates in C the bisection of S H; then $b = 0$, $b' = 0$, and $a' = -a$, and the equation becomes

$$\left(1 - \frac{a^2}{\text{A}^2}\right) x^2 + y^2 + a^2 - \text{A}^2 = 0$$

and A being $< a$, by making $a^2 - A^2 = B^2$, we have

$$-\frac{B^2}{A^2}x^2 + y^2 + B^2 = 0;$$

$$\therefore \frac{x^2}{A^2} - \frac{y^2}{B^2} = 1,$$

the simplest form of which the equation is susceptible, in which A is the focal axis.

173. Cor. 2.—Let S H be the axis of x, and the origin of co-ordinates the point A in the curve; then

$$b = 0, b' = 0,$$

$$a = SA = SC - AC = E - A,$$

$$-a' = HA = E + A;$$

$\therefore$ the equation becomes

$$\left(1 - \frac{E^2}{A^2}\right)x^2 + y^2 - \left\{2(A - E) + E\,\frac{(A - E)^2 - (A + E^2) + 4A^2}{2A^2}\right\}x = 0,$$

$$\text{or } (E^2 - A^2)\frac{x^2}{A^2} - y^2$$

$$= \frac{-4A^3 + 4A^2E - 4A^2E + 4AE^2}{2A^2}x$$

$$= 2 \cdot \frac{E^2 - A^2}{A}x;$$

$$\therefore \frac{x^2}{A^2} - \frac{y^2}{B^2} = 2 \cdot \frac{x}{A},$$

the same equation as is found directly.

174. Prop.—*The polar equation of a hyperbola, whose foci are* (o, o) *and* $(2E, \alpha)$, $2E$ *being the right line joining the foci, and* α *the traced angle of the vertex, is*

$$r = \frac{a\,(e^2 - 1)}{1 + e\cos.(\theta - \alpha)},$$

where 2a *is the axis and* $e = \frac{E}{a}$. (Fig. 47.)

Let A be the vertex, S the focus (o, o), and H the other focus $(2E, \alpha)$, SX the origin of traced angles; then P being any point (r, θ) of the curve, we have

$$\cos.(\theta - \alpha) = \cos.\,ASP = \frac{SP^2 + SH^2 - HP^2}{2SP\,.\,SH}$$

$$= \frac{r^2 + 4E^2 - (2a + r)^2}{4rE} = \frac{E^2 - a^2 - ar}{rE}.$$

But $E = ae$;

$$\therefore \cos.(\theta - \alpha) = \frac{-a^2 + a^2e^2 - ar}{aer} = \frac{a(e^2 - 1) - r}{er};$$

$$\therefore \qquad r = \frac{a\,(e^2 - 1)}{1 + e\cos.\,(\theta - \alpha)},$$

the equation required.

175. Cor.—If the origin of traced angles be SH; then $\alpha = 0$, and

$$r = \frac{a\,(e^2 - 1)}{1 + e\cos.\,\theta}.$$

176. Def.—*The* Eccentricity *of a hyperbola is the ratio of the distance between either focus and centre to that between the vertex and centre.*

Hence the eccentricity is $\frac{E}{a} = e$.

Otherwise.

The rectangular equation, when the origin is at the centre and SH the axis of y, is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$$

and proceeding as before, the equation may be shown to be

$$r = \frac{a(e^2 - 1)}{1 + e \cos.(\theta - \alpha)}.$$

177. Prop.—*The polar equation of a hyperbola whose foci are the points* $(-E, \alpha)$, $(E, \pi + \alpha)$, *is*

$$r^2 = \frac{a^2(e^2 - 1)}{e^2 \cos.^2(\theta - \alpha) - 1};$$

2 a *being the focal axis, and* e *the eccentricity.* (Fig. 48.)

This is proved, as in the case of the ellipse, by either of the methods there given.

178. Prop.—*To find the general polar equation of a hyperbola when the foci are any points* (R, α), (R', α') *whatever.* (Fig. 49.)

Let S, H, P be the points (R, α), (R', α'), (r, θ) respectively; then

$$\begin{aligned} HP^2 &= OH^2 + OP^2 - 2\,OH \,.\, OP \,.\, \cos. HOP \\ &= R'^2 + r^2 - 2R'r\cos.(\theta - \alpha') \\ SP^2 &= R^2 + r^2 - 2Rr\cos.(\theta - \alpha'); \end{aligned}$$

$$\therefore \quad (HP - SP)(HP + SP) = R'^2 - R^2$$
$$- 2\{R'\cos(\theta - \alpha') - R\cos.(\theta - \alpha)\}\,r;$$
$$\text{also} = 2a(2a + 2SP);$$

$$\therefore$$

$$SP = \frac{R'^2 - R^2 - 4a^2}{4a} - \frac{R'\cos.(\theta - \alpha') - R\cos.(\theta - \alpha)}{2a}\,r;$$

$$\therefore\ SP^2 = \left(a + \frac{R^2 - R'^2}{4a}\right)^2 - \left(a + \frac{R^2 - R'^2}{4a}\right) \times$$
$$\frac{R\cos.(\theta - \alpha) - R'\cos.(\theta - \alpha')}{2a}\, r$$
$$+ \frac{\{R\cos.(\theta - \alpha) - R'\cos.(\theta - \alpha')\}^2}{4a^2}\, r^2;$$

also $$= R^2 + r^2 - 2Rr\cos.(\theta - \alpha);$$

$$\therefore \quad \left\{1 - \frac{\{R\cos.(\theta - \alpha) - R'\cos.(\theta - \alpha')\}^2}{4a^2}\right\} r^2$$
$$-\left\{2R\cos.(\theta - \alpha) - \left(a + \frac{R^2 - R'^2}{4a}\right) \times \right.$$
$$\left.\frac{R\cos.(\theta - \alpha) - R'\cos.(\theta - \alpha')}{a}\right\} r$$
$$+ R^2 - \left(a + \frac{R^2 - R'^2}{4a}\right)^2 = 0;$$

the equation required.

179. Cor.—When $R = R'$ and $\alpha' = \pi + \alpha$; the equation becomes

$$\left\{1 - \frac{R^2}{a^2}\cos.^2(\theta - \alpha)\right\} r^2 + R^2 - a^2 = 0.$$

But here $R = ae$ and $\therefore$

$$r^2 = \frac{a^2(e^2 - 1)}{e^2\cos.^2(\theta - \alpha) - 1}.$$

This is precisely the same as the corresponding equation to the ellipse; only that e being, in the case of the hyperbola, > 1, it is proper to change all the signs.

Similarly for other cases.

180. Def.—An Asymptote to a Curve is that right

line or curve in which the *difference* of either pair of corresponding ordinates for the same origin and axes, of it and the given curves, continually decreases the further removed from the origin, and only vanishes when that distance is supposed infinite.

181. Prop.—*To find the asymptotes, rectilinear or curvilinear, of any proposed curve.*

Expanding the ordinate y by means of the equation of the curve into the descending series,

$$y = \mathrm{A}x^m + \mathrm{B}x^{m-1} + \text{\&c.} + \mathrm{M}x + \mathrm{L} + \frac{\mathrm{A}'}{x} + \frac{\mathrm{B}'}{x^2} +, \text{\&c.}$$

when this is possible; then the curve whose equation is

$$y_{\prime} = \mathrm{A}x^m + \mathrm{B}x^{m-1} + \ldots\ldots \mathrm{M}x + \mathrm{L},$$

is an asymptote to the given curve.

For $$y - y_{\prime} = \frac{\mathrm{A}}{x} + \frac{\mathrm{B}}{x^2} +, \text{\&c.};$$

and as x increases, the difference between the ordinates of the curves, viz. $y - y_{\prime}$, continually decreases, but does not vanish.

Similarly, asymptotes may be found by expanding x in a series $x = \mathrm{A}y^m +$, &c.

Ex. 1.—*To find the asymptotes, if any, of the parabola* $y^2 = 4\mathrm{S}x$.

Since $$y = \pm 2\sqrt{\mathrm{S}} \,.\, x^{\frac{1}{2}},$$

and $$x = \frac{y^2}{4\mathrm{S}},$$

no descending series can arise for the value of either co-ordinate. Consequently the parabola has no asymptotes.

182. Ex. 2.—*To find the asymptotes of a hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Since $\frac{x}{a} = \pm \left(\frac{y^2}{b^2} + 1\right)^{\frac{1}{2}}$

$$= \pm \left\{\frac{y}{b} + \tfrac{1}{2} \cdot \left(\frac{y}{b}\right)^{-1} + \tfrac{1}{2} \cdot \frac{\frac{1}{2} - 1}{2} \cdot \left(\frac{y}{b}\right)^{-} + \&c.\right\}$$

$$= \pm \left\{\frac{y}{b} + \frac{b}{2} \cdot \frac{1}{y} - \frac{b^3}{8} \cdot \frac{1}{y^3} - \&c. \right\};$$

$$\therefore \qquad \frac{x}{a} = \pm \frac{y}{b}$$

are the equations of two rectilinear asymptotes to the hyperbola.

Similarly,

$$\frac{y}{b} = \pm \left(\frac{x^2}{a^2} - 1\right)^{\frac{1}{2}} = \pm \left\{\frac{x}{a} - \tfrac{1}{2}\left(\frac{x}{a}\right)^{-1} + \tfrac{1}{8}\left(\frac{x}{a}\right)^{-3} +, \&c.\right\};$$

$$\therefore \qquad \frac{y}{b} = \pm \frac{x}{a},$$

which are the same asymptotes as before.

To construct these asymptotes (fig. 50) let $x = o$; then $y = o$, and they both pass through the origin of co-ordinates, which is the centre of the hyperbola.

Again, let $x = a$; then

$y = \pm b$, and if C A $= a$ and A B $= b$, and A B′ $= - b$;

then C B, C B′ are the asymptotes of the hyperbola.

183. Ex. 3.—*To find the asymptotes of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = - 2 \cdot \frac{x}{a}$.

Since $\frac{y}{b} = \pm \left(\frac{x^2}{a^2} + 2 \cdot \frac{x}{a}\right)^{\frac{1}{2}}$

$$= \pm \frac{x}{a}\left(1 + 2\frac{a}{x}\right)^{\frac{1}{2}}$$

$$= \pm \frac{x}{a}\left(1 + \frac{a}{x} + \tfrac{1}{2} \cdot \frac{\frac{1}{2} - 1}{2} \cdot \frac{4a^2}{x^2} -, \&c.\right)$$

$$= \pm \left(\frac{x}{a} + 1 - \tfrac{1}{2}\frac{a}{x} +, \&c.\right);$$

$\therefore$ the asymptotes are the right lines

$$\frac{y}{b} = \pm \left(\frac{x}{a} + 1\right);$$

that is, the right lines

$$\left.\begin{aligned} \frac{x}{-a} + \frac{y}{b} &= 1 \\ \text{and } \frac{x}{-a} + \frac{y}{-b} &= 1 \end{aligned}\right\};$$

$\therefore$ their common ordinates with the axes of x and y, are respectively

$$-a, b \text{ and } -a \text{ and } -b;$$

whence they are easily constructed as before.

It is evident the ellipse cannot have any rectilinear asymptote, because of its *finite* figure.

184. Prop.—*To find the equation of the hyperbola, the axes of co-ordinates being its asymptotes.* (Fig. 51.)

Let C X, C Y be the asymptotes, which meet at the centre C; and let S and H be the foci; then $HP^2 = HD^2 + PD^2 - 2HD.PD.\cos.HPD$, P being any point in the hyperbola, and P M, C M its oblique co-ordinates y and x respectively.

But, calling $\angle$ Y C S $= \alpha$,

$$\text{H D} = \text{H C} + \text{C D} = \text{E} + \text{C M} \,.\, \frac{\sin.\, 2\,\alpha}{\sin.\, \alpha}$$

$$= \text{E} + 2x \,.\, \cos.\, \alpha,$$

and $\text{P D} = y - \text{M D} = y - \text{C M} = y - x$;

$$\therefore \quad \text{H P}^2 = (\text{E} + 2x \cos.\, \alpha)^2 + (y - x)^2$$
$$+ 2\,(\text{E} + 2x \cos.\, \alpha)\,(y - x) \cos.\, \alpha$$
$$= (y - x)^2 + 4xy \cos.^2 \alpha$$
$$+ 2\,\text{E}\,(x + y) \cos.\, \alpha + \text{E}^2.$$

Similarly, $\text{S P}^2 = (y - x)^2 + 4xy \cos.^2 \alpha$

$$- 2\,\text{E}\,(x + y) \cos.\, \alpha + \text{E}^2;$$

$$\therefore \quad \text{H P}^2 - \text{S P}^2 = 4\,\text{E}\,(x + y) \cos.\, \alpha,$$
$$= \text{also } 4a\,(a + \text{S P});$$

$$\therefore \qquad \text{S P} = \frac{\text{E}}{a}\,(x + y) \cos.\, \alpha - a;$$

$$\therefore \quad \text{S P}^2 = \frac{\text{E}^2}{a^2}(x + y)^2 \cos.^2 \alpha - 2\,\text{E}\,(x + y) \cos.\, \alpha + a^2;$$

$$\text{also} = (x - y)^2 + 4\,xy \cos.^2 \alpha - 2\,\text{E}\,(x + y) \cos.\, \alpha + \text{E}^2;$$

$$\therefore \cos.^2 \alpha \left\{\frac{a^2 + b^2}{a^2} \,.\, (x + y)^2 - 4\,xy\right\} - (x - y)^2 = b^2.$$

But, $$\cos.^2 \alpha = \frac{\text{C A}^2}{\text{C B}^2} = \frac{a^2}{a^2 + b^2};$$

$$\therefore \quad (x + y)^2 - (x - y)^2 - 4\,\frac{a^2}{a^2 + b^2} \,.\, xy = b^2,$$

or $$4\,xy \,.\, \frac{b^2}{a^2 + b^2} = b^2;$$

$$\therefore \qquad xy = \frac{a^2 + b^2}{4},$$

which is a direct investigation from the very definition of the curve.

It would be briefer to deduce this equation from the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, or from $\frac{x^2}{a^2} + 2 \cdot \frac{x}{a} = \frac{y^2}{b^2}$; but then it would be necessary to take for granted several properties of the asymptote and hyperbola.

185. Def.—The Equilateral Hyperbola is that in which $a = b$; or in which $E^2 = a^2 + b^2 = 2a^2$.

186. Cor.—The equation of an equilateral hyperbola referred to the asymptotes is

$$xy = \frac{a^2}{2}.$$

Recapitulation of the Equations of the Hyperbola.

Rectangular Equations.

187.—1. When the foci are both in the axis of x and the origin the bisection of the distance between them, the equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

in which $2a$ is the axis, and $b = \sqrt{(E^2 - a^2)}$.

2. When the foci are both in the axis of x and the origin of co-ordinates in the curve, the equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -2 \cdot \frac{x}{a}.$$

3. When the foci are both in the axis of x and the origin of co-ordinates one of the foci, the equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2\sqrt{(a^2 + b^2)}}{a^2}\, x - \frac{b^2}{a^2}.$$

4. The general equation is the same as that for the ellipse.

Polar Equations.

5. When the foci are the points (o, α), $(2\text{E}, \pi + \alpha)$; or when the pole is in the focus, the equation is

$$r = \frac{a(e^2 - 1)}{1 + e \cos.(\theta - \alpha)}.$$

6. When the foci are the points (E, α), $(\text{E}, \pi + \alpha)$; or when the pole is in the centre, the equation is

$$r^2 = \frac{a^2 \cdot (e^2 - 1)}{e^2 \cos.^2 \theta - 1}.$$

7. The general polar equation is the same as that of the ellipse.

8. The equation of a hyperbola whose axes of coordinates are its asymptotes, is

$$x\,y = \frac{a^2 + b^2}{4}.$$

9. The equation of an equilateral hyperbola, the axes of co-ordinates being its asymptotes, is

$$x\,y = \frac{a^2}{2}.$$

Problems on Right Lines, Circles, Parabolas, Ellipses, and Hyperbolas.

The common points of a hyperbola with the co-ordinate axes, when anywhere situated with respect to those axes, may be found exactly as for the ellipse; as also those of a right line and hyperbola; those of a circle and hyperbola; those of a parabola and hyperbola; and those of an ellipse and hyperbola.

The process is the same exactly in all the problems that we have given upon the ellipse, and the results also often the same, scarcely ever being different, except

in sign. This being the case, it is not of importance to repeat all those problems. The more important of them, however, we shall sketch as concisely as we can.

188. PROB.—*To find the equation of the locus of the extremity of the right line drawn from any point of a hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, *parallel to the focal axis and in a given ratio* (n) *to the distance of that point from the focus.* (Fig. 52.)

Proceeding as for the ellipse, we shall have

$$\frac{\text{S P}}{\text{P P}'} = n$$

$$\begin{aligned} y'^2 &= \text{PM}^2 = \text{S P}^2 - \text{S M}^2 \\ &= n^2 . \text{P P}'^2 - \text{S M}^2 \\ &= n^2 . (x' - x)^2 - (\text{E} - x)^2 ; \end{aligned}$$

also, by the hyperbola,

$$\frac{x^2}{a^2} - \frac{y'^2}{b^2} = 1 ;$$

$$\therefore \qquad x = a \sqrt{\left(1 + \frac{y'^2}{b^2}\right)} ;$$

$$\therefore\ y'^2 = n^2 \left\{x' - a\sqrt{\left(1 + \frac{y'^2}{b^2}\right)}\right\}^2 - \left\{\text{E} - a\sqrt{\left(1 + \frac{y'^2}{b^2}\right)}\right\}$$

whence

$$n\left\{x' - a\sqrt{\left(1 + \frac{y'^2}{b^2}\right)}\right\} = \pm a \mp \text{E} \sqrt{\left(1 + \frac{y'^2}{b^2}\right)},$$

or $$x' = \pm \frac{a}{n} + \left(a \mp \frac{\text{E}}{n}\right) \sqrt{\left(1 + \frac{y'^2}{b^2}\right)} ;$$

which two equations indicate two loci, one generated by drawing P P′ on the left of P, and the other on the right.

But these equations being rationalized become

$$\frac{\left(x' \mp \frac{a}{n}\right)^2}{\left(a \mp \frac{E}{n}\right)^2} - \frac{y'^2}{b^2} = 1;$$

$\therefore$ the loci required are hyperbolas whose centres are $\left(\frac{a}{n}, o\right)$, $\left(-\frac{a}{n}, o\right)$ and semi-focal axes $\frac{E}{n} - a$, and $\frac{E}{n} + a$.

189. Cor. 1.—When $n = 1$, these loci are the hyperbolas

$$\frac{(x - a)^2}{(a - E)^2} - \frac{y^2}{b^2} = 1,$$

and
$$\frac{(x + a)^2}{(a + E)^2} - \frac{y^2}{b^2} = 1,$$

whose centres are (a, o), $(-a, o)$ and semi-focal axes $E - a$, and $E + a$; which are thus represented in the diagram (53); being two hyperbolas which both pass through the focus.

190. Cor. 2.—When $E = 0$ or $a = b$, the given hyperbola is equilateral, and so are the loci.

191. Cor. 3.—When $a \mp \frac{E}{n} = 0$ or $n = \pm \frac{E}{a}$;

then
$$x' = \pm \frac{a^2}{E} = \pm \frac{a^2}{\sqrt{(a^2 + b^2)}}$$

the equations of two right lines which are at right angles to the axis of x and between the vertices and the centre.

These, as in the ellipse, have been defined by the geometrical writers and called

DIRECTRIX.

Observe; in this case n is $>$ unit, and $\therefore$ S P is $>$ PP$'$.

192. PROB.—*The equation of the tangent at any point* (a, b) *of the hyperbola* $\frac{x^2}{a'} - \frac{y^2}{b'^2} = 1$ is

$$\frac{x}{\left(\frac{a'^2}{a}\right)} - \frac{y}{\left(\frac{b'^2}{b}\right)} = 1.$$

When the hyperbola is equilateral or $a' = b'$; it is

$$\frac{x}{\left(\frac{a'^2}{a}\right)} - \frac{y}{\left(\frac{b'^2}{b}\right)} = 1.$$

A corollary similar to that for the ellipse may also be deduced, and in the same way.

193. PROB.—*The equation of the normal at any point* (a, b) *of a hyperbola* $\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1$ *is*

$$\frac{x}{a\left(1 + \frac{b'^2}{a'^2}\right)} + \frac{y}{b\left(1 + \frac{a'^2}{b'^2}\right)} = 1.$$

For the equilateral hyperbola it is

$$\frac{x}{2a} + \frac{y}{2b} = 1.$$

194. PROB.—*The equation between the perpendicular* p, *from the focus to the tangent at the point* (a, b) *of a hyperbola* $\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1$, *and the radius vector* r, *is*

$$p^2 = \frac{b'^2 r}{2a' + r}.$$

195. Prob.—*The equation between the perpendicular* p, *from the centre to the tangent at the point* (a, b) *of a hyperbola* $\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1$ *and the radius vector, the pole being the centre, is*

$$p^2 = \frac{a'^2\, b'^2}{r^2 - a'^2 + b'^2}.$$

196. Prob.—*The circle of curvature at any point* (a, b) *of a hyperbola* $\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1$ being assumed to be

$$(x - \mathrm{A})^2 + (y - \mathrm{B})^2 = \rho^2$$

in which (A, B) is the centre and ρ the radius, by pursuing the same process as in the case of the ellipse, we shall find that

$$\mathrm{A} = \frac{\mathrm{E}^2\, a^3}{a'^4}$$

$$\mathrm{B} = \frac{\mathrm{E}^2\, b^3}{b'^4}$$

$$\text{and } \rho = \frac{(a^2 + b^2 - a'^2 + b'^2)^{\frac{3}{2}}}{a'\, b'}.$$

197. Prob.—*The chord of the osculating circle at the point* (a, b) *of a hyperbola,* $\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1$, *which passes through that point and the centre of the hyperbola, is*

$$2 . \frac{a^2 + b^2 - a'^2 + b'^2}{\sqrt{(a^2 + b^2)}}.$$

198. Prob.—*The chord of the osculating circle at the point* (a, b) *of a hyperbola* $\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1$, *which passes through* (a, b) *and the focus* $(-\mathrm{E}, o)$ *is*

$$2 \cdot \frac{a^2 + b^2 - a'^2 + b'^2}{a'}.$$

199. PROB.—*The equation of the evolute of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b'^2} = 1$ *is*

$$\left(\frac{a' x}{a'^2 + b'^2}\right)^{\frac{2}{3}} - \left(\frac{b' y}{a'^2 + b'^2}\right)^{\frac{2}{3}} = 1$$

SECTION VII.

THEORY OF THE GENERAL CONIC SECTION.

200. PROP.—*The equations of all the Conic Sections of a right line, a circle, a parabola, an ellipse, and a hyperbola, may be deduced from that of one of them;* viz. *that of the ellipse.*

Taking $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 \cdot \frac{x}{a}$

for the equation of the ellipse, the vertex of which is the origin of co-ordinates and axis of x the focal axis, we have

1. When $b = \infty$ and a is finite

$$x^2 = 2 a x$$

$$\text{and } x = 0, x = 2a$$

are the equations of two right lines, the axis of y and the right line parallel to it at the other vertex.

2. When $b = a$, the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 2 \frac{x}{a}$$

the equation of a circle.

3. When $a = \infty$ and the focal distance S from the vertex is finite; we have

$$b^2 = a^2 - E^2 = (a - E) \cdot (a + E) = S \cdot 2a$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{2Sa} = 2 \cdot \frac{x}{a}$$

$$\therefore \frac{x^2}{a} + \frac{y^2}{2S} = 2x$$

and a being $= \infty$, $\frac{x^2}{a} = 0$;

$\therefore y^2 = 4 S x$ the equation of a parabola.

4. When E is $> a$, $b^2 = a^2 - E^2$ is negative,

and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2 \frac{x}{a}$

the equation of a hyperbola.

Hence it follows that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 \frac{x}{a}$$

is the equation of a right line, a circle, a parabola, an ellipse, or hyperbola, according as b *is* o, *equal to* a, a $= \infty$, b *unequal to* a, *or* b *is imaginary.*

It is, therefore, the general equation of a Conic Section, whose origin of co-ordinates is the vertex and axis of x the focal axis.

The general polar equation may easily be found by substitution.

201. Prop.—*To find the equation of the tangent at any point* (a', b') *of the general conic section*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 \frac{x}{a}.$$

Let the equation of the secant passing through (a', b'), (a'', b'') of the conic section be

$$\frac{x}{a'''} + \frac{y}{b'''} = 1;$$

then (see p. 61) $a''' = a' - b' . \frac{a' - a''}{b' - b''}$

But $\left.\begin{aligned} \frac{a'^2}{a^2} + \frac{b'^2}{b^2} = 2 . \frac{a'}{a} \\ \frac{a''^2}{a^2} + \frac{b''^2}{b^2} = 2 . \frac{a''}{a} \end{aligned}\right\}$

$$\therefore \frac{a' - a''}{b' - b''} = - \frac{a^2}{b^2} . \frac{b' + b''}{a' + a'' - 2a}$$

$$\therefore a''' = a' + b' . \frac{a^2}{b^2} . \frac{b' + b''}{a' + a'' - 2a} .$$

Also $b''' = b' - a' . \frac{b' - b''}{a' - a''}$

$$= b' + a' . \frac{b^2}{a^2} . \frac{a' + a'' - 2a}{b' + b''} .$$

Let the secant become the tangent, or $a'' = a'$, $b'' = b'$ and we have

$$a''' = a' + \frac{b'^2 a^2}{b^2(a' - a)} = \frac{a'^2 b^2 + a^2 b'^2 - a a' b^2}{(a' - a) b^2}$$

But $a'^2 b^2 + a^2 b'^2 = 2 a a' b^2$

$$\therefore a''' = \frac{a a'}{a' - a} .$$

Also $b''' = b' + a' . \frac{b^2}{a^2} . \frac{a' - a}{b'}$

$$= \frac{a'^2 b'^2 + a'^2 b^2 - a a' b^2}{a^2 b'}$$

$$= \frac{a a' b^2}{a^2 b'} = \frac{a' b^2}{a b'}$$

$\therefore$ the equation required is

$$\frac{x}{\left(\frac{a'a}{a'-a}\right)} + \frac{y}{\left(\frac{a'b^2}{ab'}\right)} = 1.$$

202. Prop.—*To find the equation of the normal at any (a', b') of the General Conic Section*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2\frac{x}{a}.$$

Let $\frac{x}{a''} + \frac{y}{b''} = 1$ be the tangent,

and $\frac{x}{a'''} + \frac{y}{b'''} = 1$ the normal,

then $a''' = a' - b' \cdot \frac{b''}{a''}$, $b''' = b' - a' \cdot \frac{a''}{b''}$.

Also $a'' = \frac{a'a}{a'-a}$ and $b'' = \frac{a'b^2}{ab'}$

$$\therefore \quad \frac{a''}{b''} = \frac{a'a}{a'-a} \times \frac{ab'}{a'b^2} = \frac{a^2b'}{(a'-a)b^2}$$

$$\therefore \quad a''' = a' - (a'-a)\frac{b^2}{a^2}$$

$$b''' = b' - \frac{a^2}{b^2} \cdot \frac{a'b'}{a'-a},$$

$\therefore$ the required equation is

$$\frac{x}{a' - (a'-a)\frac{b^2}{a^2}} + \frac{y}{b' - \frac{a'b'}{a'-a} \cdot \frac{a^2}{b^2}} = 1.$$

203. Prop.—*To find the equation of the osculating circle at the point (a, b) of the general Conic Section*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 \cdot \frac{x}{a}.$$

Let $(x - A)^2 + (y - B)^2 = \rho^2$ be the equation of

the secant circle of the conic section, passing through the points (a, b), $(a_{,}, b_{,})$, $(a_{,,}, b_{,,})$ in which A, B, ρ are to be determined on the supposition that the points $(a_{,}, b_{,})$, $(a_{,,}, b_{,,})$ coincide with the point $(a\,b)$; or that the secant circle becomes the osculating circle.

Since the secant circle passes through (a, b), $(a_{,}, b_{,})$ $(a_{,,}, b_{,,})$, we have

$$\begin{aligned}(a \;\; - \mathrm{A})^2 + (b \;\; - \mathrm{B})^2 &= \rho^2\\ (a_{,} - \mathrm{A})^2 + (b_{,} - \mathrm{B})^2 &= \rho^2\\ (a_{,,} - \mathrm{A})^2 + (b_{,,} - \mathrm{B})^2 &= \rho^2\end{aligned}$$

which by eliminating ρ, give

$$\begin{aligned}2\,\mathrm{A}\,(a - a_{,}) + 2\,\mathrm{B}\,(b - b_{,}) &= a^2 - a_{,}^2 + b^2 - b_{,}^2\\ 2\,\mathrm{A}\,(a - a_{,,}) + 2\,\mathrm{B}\,(b - b_{,,}) &= a - a_{,,}^2 + b^2 - b_{,,}^2.\end{aligned}$$

But also, since (a, b), $(a_{,}, b_{,})$, $(a_{,,}, b_{,,})$ are points of the conic section, we have

$$\left.\begin{aligned}\frac{a^2}{a'^2} + \frac{b^2}{b'^2} &= 2\,.\,\frac{a}{a'}\\ \frac{a_{,}^2}{a'^2} + \frac{b_{,}^2}{b'^2} &= 2\,.\,\frac{a_{,}}{a'}\\ \frac{a_{,,}^2}{a'^2} + \frac{b_{,,}^2}{b'^2} &= 2\,.\,\frac{a_{,,}}{a'}\end{aligned}\right\};$$

$$\therefore\quad \left.\begin{aligned}\frac{a^2 - a_{,}^2}{a'^2} - 2\,.\,\frac{a - a_{,}}{a'} &= -\frac{b^2 - b_{,}^2}{b'^2}\\ \frac{a^2 - a_{,,}^2}{a'^2} - 2\,.\,\frac{a - a_{,,}}{a'} &= -\frac{b^2 - b_{,,}^2}{b'^2}\\ \frac{a_{,}^2 - a_{,,}^2}{a'^2} - 2\,.\,\frac{a_{,} - a_{,,}}{a'} &= -\frac{b_{,}^2 - b_{,,}^2}{b'^2}\end{aligned}\right\}.$$

From the former reduced equations we get

$$\left.\begin{array}{l}\dfrac{a-a_{,}}{b-b_{,}}(2\,\mathrm{A}-a-a_{,}) = -(2\,\mathrm{B}-b-b_{,}) \\ \dfrac{a-a_{,,}}{b-b_{,,}}(2\mathrm{A}-a-a_{,,}) = -(2\,\mathrm{B}-b-b_{,,})\end{array}\right\};$$

also from the latter

$$\left.\begin{array}{l}\dfrac{a-a_{,}}{b-b_{,}}(a+a_{,}-2\,a') = -\dfrac{a'^2}{b'^2}\cdot(b+b_{,}) \\ \dfrac{a-a_{,,}}{b-b_{,,}}(a+a_{,,}-2\,a') = -\dfrac{a'^2}{b'^2}(b+b_{,,})\end{array}\right\}.$$

Hence eliminating $\dfrac{a-a_{,}}{b-b_{,}}$ and $\dfrac{a-a_{,,}}{b-b_{,,}}$, we get

$$\frac{2\,\mathrm{A}-a-a_{,}}{2\,a'-a-a'} = -\frac{b'^2}{a'^2\,(b+b_{,})}\cdot(2\,\mathrm{B}-b-b_{,}),$$

$$\frac{2\,\mathrm{A}-a-a_{,,}}{2\,a'-a-a_{,,}} = -\frac{b'^2}{a'^2(b+b_{,,})}(2\,\mathrm{B}-b-b_{,,}),$$

$$\therefore\ 2\,\mathrm{A}-a-a_{,} = -\frac{2\,a'-a-a_{,}}{b+b_{,}}\cdot\frac{b'^2}{a'^2}(2\mathrm{B}-b-b_{,}),$$

and $2\,\mathrm{A}-a-a_{,,} = -\dfrac{2\,a'-a-a_{,,}}{b+b_{,,}}\cdot\dfrac{b'^2}{a'^2}\cdot(2\mathrm{B}-b-b_{,,})$;

$\therefore$ by subtraction

$$a_{,,}-a_{,} = 2\frac{b'^2}{a'^2}\,\mathrm{B}\cdot\left(\frac{2\,a'-a-a_{,,}}{b+b_{,,}} - \frac{2\,a'-a-a_{,}}{b+b_{,}}\right)$$
$$+\ (2\,a'-a-a_{,}-2\,a'+a+a_{,,})\,\frac{b'^2}{a'^2};$$

$$\therefore\ 2\,\frac{b'^2}{a'^2}\mathrm{B}\,.\,\frac{(2\,a'-a)(b_{,}-b_{,,})+b(a_{,}-a_{,,})-(a_{,,}b_{,}-a_{,}b_{,,})}{(b+b_{,})\,(b+b_{,,})} =$$
$$-\,(a_{,}-a_{,,})\cdot\left(1-\frac{b'^2}{a'^2}\right),$$

$$\therefore\ 2\,b'^2\,\mathrm{B}\,.\left\{(2\,a'-a)\frac{b_{,}-b_{,,}}{a_{,}-a_{,,}}+b+\frac{a_{,}b_{,,}-a_{,,}b_{,}}{a_{,}-a_{,,}}\right\}=$$
$$-\,(b+b_{,})\,(b+b_{,,})\,.\,(a'^2-b'^2).$$

But $$\frac{a_{,}b_{,,}-a_{,,}b_{,}}{a_{,}-a_{,,}}=b_{,,}-a_{,,}\frac{b_{,}-b_{,,}}{a_{,}-a_{,,}},$$

and $$\frac{b_{,}-b_{,,}}{a_{,}-a_{,,}}=-\frac{b'^2}{a'^2}\ \frac{a_{,}+a_{,,}-2\,a'}{b_{,}+b_{,,}};$$

$$\therefore\ 2b'^2\mathrm{B}\left\{-(2\,a'-a-a_{,,})\,\frac{b'^2}{a'^2}\,.\,\frac{a_{,}+a_{,,}-2a'}{b_{,}+b_{,,}}+b+b_{,,}\right\}=$$
$$-\,(b+b_{,})\,(b+b_{,,})\,(a'^2-b'^2)\,.$$

Let the secant circle become the osculating circle; that is, let $a_{,}=a_{,,}=a$, $b_{,}=b_{,,}=b$, and

$$2\,b'^2\,\mathrm{B}\left\{(2\,a'-2\,a)\,.\,\frac{b'^2}{a'^2}.\ \frac{2\,a-2\,a'}{2\,b}-2\,b\right\}$$
$$=4\,b^2\,(a'^2-b'^2)$$
$$\therefore\ -\,\mathrm{B}\left\{\frac{(a'-a)^2}{b}\,.\,\frac{b'^2}{a'^2}+b\right\}=\frac{b^2}{b'^2}\,(a'^2-b'^2)\,,$$
$$\therefore\ \mathrm{B}=-\,\frac{a'^2}{b'^2}\,b^3\,.\,\frac{a'^2-b'^2}{a'^2\,b^2+(a'-a)^2\,b'^2}.$$

Also, when the secant becomes the osculating circle

$$\frac{\mathrm{A}-a}{a'-a}=-\,\frac{b'^2}{a'^2\,b}\,.\,(\mathrm{B}-b),$$

whence A; and A being known ρ is found from

$$\rho^2=(\mathrm{A}-a)^2+(\mathrm{B}-b)^2$$

which will complete the operation.

It has been shown that there is, for each of the Conic Sections, a right line, called the Directrix, so situated that the distance of every point in the Conic Section from the focus is in a given ratio to its perpendicular

distance upon that right line. It hence appears that I might have defined Conic Sections thus :

DEF.—A CONIC SECTION *is a plane curve, the distance of every point of which from a given point in that plane is to the perpendicular drawn from it upon a given right line in that plane, in a given ratio.*

Many writers have adopted this definition. But quite as many others, that of the sum and difference of the focal distances from any point of the Conic Section being constant. Both these definitions are derivable from each other, and also from the properties of the sections of a right cone. That nothing, however, may be wanting in this treatise, I add to this division of the subject the following general proposition, founded upon he preceding definition.

204. PROP.—*To find the equation of the Conic Section whose directrix is the right line* $\frac{x}{a} + \frac{y}{b} = 1$, *and whose focus is* (a', b'). (Fig. 54.)

Let the directrix be the right line A B, S the focus, and P any point (x, y) of the curve.

Also P N $\perp$ A B, &c., as is obvious.

Since the ratio of S P to P N is given ; let

$$\frac{\mathrm{SP}}{\mathrm{PN}} = n;$$

then PN being $\perp$ to $\frac{x}{a} + \frac{y}{b} = 1$, we have (Art. 51.)

$$\mathrm{PN} = \frac{bx + ay - ab}{\sqrt{(a^2 + b^2)}};$$

also $SP^2 = (x - a')^2 + (y - b')^2$;

$$\therefore \quad (x - a')^2 + (y - b')^2 = n^2 . \frac{(bx + ay - ab)^2}{a^2 + b^2};$$

$$\therefore \quad (x^2 + y^2 - 2a'x - 2b'y + a'^2 + b'^2) \frac{a^2 + b^2}{n^2}$$

$$= b^2x^2 + a^2y^2 + 2abxy - 2ab^2x - 2a^2by + a^2b^2;$$

$$\therefore \left(\frac{a^2 + b^2}{n^2} - b^2\right)x^2 + \left(\frac{a^2 + b^2}{n^2} - a^2\right)y^2$$

$$- 2abxy + 2\left(ab^2 - a'.\frac{a^2 + b^2}{n^2}\right)x$$

$$+ 2\left(a^2b - b'.\frac{a^2+b^2}{n^2}\right)y + \frac{(a^2+b^2)(a'^2+b'^2)}{n^2} - a^2b^2 = 0,$$

which is the equation of every conic section, however it may be situated in a given plane.

205. Cor. 1.—Let the focus be in the axis of x; then $b' = 0$, and the equation becomes

$$\left(\frac{a^2 + b^2}{n^2} - b^2\right)x^2 + \left(\frac{a^2 + b^2}{n^2} - a^2\right)y^2 - 2abxy$$

$$+ 2\left(ab^2 - a'.\frac{a^2 + b^2}{n^2}\right)x + 2a^2by + \frac{(a^2 + b^2)a'^2}{n^2}$$

$$- a^2b^2 = 0.$$

206. Cor. 2.—Let the focus be in the axis of x and the directrix parallel to y; then $b' = 0$ and $b = \infty$, and the equation becomes

$$\left(\frac{1}{n^2} - 1\right)x^2 + \frac{y^2}{n^2} + 2\left(a - \frac{a'}{n^2}\right)x + \frac{a'^2}{n^2} - a^2 = 0.$$

207. Cor. 3.—Let the directrix and focus be on different sides of the axis of y, and so placed that the curve

shall pass through the origin, the focus being in the axis of x and the directrix parallel to that of y; that is, let

$$b' = 0,\ b = \infty\text{, and } \frac{a'}{-a} = n;$$

then $$(1 - n^2)\,x^2 + y^2 - 2\,a'\,(n + 1)\,x = 0.$$

$$\therefore \quad \frac{x^2}{\left(\frac{a'}{1-n}\right)^2} + \frac{y^2}{\left(\frac{1+n}{1-n}a'^2\right)} = 2 \cdot \frac{x}{\left(\frac{a'}{1-n}\right)},$$

which is the equation to a general conic section, in which the semi-focal and semi-non-focal axes are

$$\frac{a'}{1-n} \text{ and } a' \sqrt{\frac{1+n}{1-n}}.$$

and of which the vertex is the origin, and focal-axis the axis of x.

When $n = 1$; this becomes

$y^2 = 4\,a'\,x$ the parabola.

When n is < 1, $1 - n$ is positive, and the section is an ellipse.

When n is > 1, $1 - n$ is negative, and the section is a hyperbola.

When $$\frac{1+n}{1-n} = \frac{1}{(1-n)^2},$$

or, $1 - n^2 = 1$ or $n = 0$,

then $$\frac{x^2}{a'^2} + \frac{y^2}{a'^2} = 2 \cdot \frac{x}{a'},$$

the equation of a circle.

208. Cor. 4.—Let $b' = 0$, $b = \infty$ as before, and $a = a' + d$, d being indeterminate; then

$$\left(\frac{1}{n^2} - 1\right)x^2 + \frac{y^2}{n^2} + 2\left(a' + d - \frac{a'}{n^2}\right)x$$
$$+ \frac{a'^2}{n^2} - (a' + d)^2 = 0.$$

Assume $a' + d - \frac{a'}{n^2} = 0;$

$$\therefore \quad d = a'\left(\frac{1}{n^2} - 1\right),$$

and $$a' + d = \frac{a'}{n^2},$$

and the equation becomes

$$\left(\frac{1}{n^2} - 1\right)x^2 + \frac{y^2}{n^2} = \frac{a'^2}{n^4} - \frac{a'^2}{n^2}$$
$$= \frac{a'^2}{n^4} \cdot (1 - n^2),$$

$$\therefore \quad \frac{x^2}{\left(\frac{a'^2}{n^2}\right)} + \frac{y^2}{\frac{a'^2(1 - n^2)}{n^2}} = 1,$$

which gives a circle, parabola, ellipse, or hyperbola according as

$1 - n^2 = 1$, $1 - n^2 = 0$, $1 - n^2$ be positive, or $1 - n^2$ is negative,

that is, according as

$n = 0$, $n = 1$, $n < 1$, or n is > 1.

209. Cor. 5.—It appears by the proposition that the form of the equation of the general Conic Section is or may be reduced to

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{xy}{C^2} + \frac{x}{D} + \frac{y}{E} + 1 = 0,$$

which is a general equation of two dimensions with respect to x and y the co-ordinates.

210. Cor. 6.—Hence, conversely, *Every equation of two dimensions with respect to two variables* x, y, *is the equation of some one of the conic sections.*

For if $\frac{x}{a} + \frac{y}{b} = 1$ be the equation of the directrix, and the focus be the point (a', b') of a conic section, then by the definition, &c.

$$(x - a')^2 + (y - b')^2 = n^2 \cdot \frac{(bx + ay - ab)^2}{a^2 + b^2};$$

which, as we have seen, is reducible to

$$\left(\frac{a^2 + b^2}{n^2} - b^2\right)x^2 + \left(\frac{a^2 + b^2}{n^2} - a^2\right)y^2 - 2abxy$$

$$+ 2\left(ab^2 - a' . \frac{a^2 + b^2}{n^2}\right) x$$

$$+ 2\left(a^2 b - b' . \frac{a^2 + b^2}{n^2}\right) y$$

$$+ \frac{(a^2 + b^2)(a'^2 + b'^2)}{n^2} - a^2 b^2 = 0 \;.. (a)$$

Now supposing that we have given a general equation of two dimensions with respect to x and y, its simplest form will be

$$Ax^2 + By^2 + Cxy + Dx + Ey + 1 = 0,$$

which, whatever the co-efficients may be, is easily reduced to the homogeneous form

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{xy}{C^2} + \frac{x}{D} + \frac{y}{E} + 1 = 0,$$

and supposing this given equation to be identical with the assumed indeterminate one of conic sections, we have by equating co-efficients,

$$\left.\begin{aligned}
\frac{a^2+b^2}{n^2}-b^2 &= \left\{\frac{a^2+b^2}{n^2}\cdot(a'^2+b'^2)-a^2b^2\right\}\frac{1}{A^2}\\
\frac{a^2+b^2}{n^2}-a^2 &= \left\{\qquad\qquad\right\}\cdot\frac{1}{B^2}\\
-2ab &= \left\{\qquad\qquad\right\}\cdot\frac{1}{C^2}\\
2\left(ab^2-a'\cdot\frac{a^2+b^2}{n^2}\right) &= \left\{\qquad\qquad\right\}\cdot\frac{1}{D}\\
2\left(a^2b-b'\cdot\frac{a^2+b^2}{n^2}\right) &= \left\{\qquad\qquad\right\}\cdot\frac{1}{E}
\end{aligned}\right\}$$

from which may easily be found a, a', b, b' and n.

First they are reducible to

$$\left.\begin{aligned}
\frac{a^2+b^2}{n^2}\cdot\left(1-\frac{a'^2+b'^2}{A^2}\right) &= b^2-\frac{a^2b^2}{A^2}\\
\frac{a^2+b^2}{n^2}\cdot\left(1-\frac{a'^2+b'^2}{B^2}\right) &= a^2-\frac{a^2b^2}{B^2}\\
\frac{a^2+b^2}{n^2} &= \frac{a^2b^2-2abC^2}{a'^2+b'^2}\\
\frac{a^2+b^2}{n^2}\left(\frac{a'^2+b'^2}{D}+2a'\right) &= \frac{a^2b^2}{D}+2ab^2\\
\frac{a^2+b^2}{n^2}\left(\frac{a'^2+b'^2}{E}+2b'\right) &= \frac{a^2b^2}{E}+2a^2b
\end{aligned}\right\}$$

$$\therefore\quad\left.\begin{aligned}
\frac{a^2b-2aC^2}{a'^2+b'^2}\left(1-\frac{a'^2+b'^2}{A^2}\right) &= b-\frac{a^2b}{A^2}\\
\frac{ab^2-2bC^2}{a'^2+b'^2}\left(1-\frac{a'^2+b'^2}{B^2}\right) &= a-\frac{ab^2}{B^2}\\
\frac{ab-2C^2}{a'^2+b'^2}\left(\frac{a'^2+b'^2}{D}+2a'\right) &= \frac{ab}{D}+2b\\
\frac{ab-2C^2}{a'^2+b'^2}\left(\frac{a'^2+b'^2}{E}+2b'\right) &= \frac{ab}{E}+2a
\end{aligned}\right\}\cdot\cdot(A)$$

$$\left.\begin{aligned}
\text{or}\quad \frac{a^2 b - 2aC^2}{a'^2 + b'^2} &= b - \frac{2aC^2}{A^2}\\
\frac{ab^2 - 2bC^2}{a'^2 + b'^2} &= a - \frac{2bC^2}{B^2}\\
\frac{a'}{a'^2 + b'^2}(ab - 2C^2) &= \frac{C^2}{D} + b\\
\frac{b'}{a'^2 + b'^2}(ab - 2C^2) &= \frac{C^2}{E} + a
\end{aligned}\right\} \quad \ldots\ldots \quad (B)$$

From the two former of equations B, we get

$$(a^2 b - 2aC^2)\left(a - 2b\frac{C^2}{B^2}\right)$$

$$= (ab^2 - 2bC^2)\left(b - 2a\frac{C^2}{A^2}\right);$$

which is easily reduced to

$$\left\{a^2 - b^2 + 2\left(\frac{C^2}{A^2} - \frac{C^2}{B^2}\right)ab\right\}(ab - 2C^2) = 0 \ldots (C.$$

Also, squaring each of the two last of equations (B) and adding the result, we get

$$\frac{(ab - 2C^2)^2}{a'^2 + b'^2} = \left(\frac{C^2}{D} + b\right)^2 + \left(\frac{C^2}{E} + a\right)^2.$$

Hence, from the first of equations (B),

$$a\left\{\left(\frac{C^2}{D} + b\right)^2 + \left(\frac{C^2}{E} + b\right)^2\right\}$$

$$= (ab - 2C^2)\left(b - 2a\frac{C^2}{A^2}\right).$$

Now $ab - 2C^2 = 0$ cannot be simultaneous with this equation, whose first member is a positive quantity; we therefore get

$$\left.\begin{aligned} a^2 - b^2 + 2\cdot\left(\frac{C^2}{A^2} - \frac{C^2}{B^2}\right) a\, b = 0 \\ \left(a + \frac{C^2}{E}\right)^2 + \left(b + \frac{C^2}{D}\right)^2 = (a\,b - 2\,C^2)\left(\frac{b}{a} - 2\frac{C^2}{A^2}\right) \end{aligned}\right\}(D)$$

from which equations a and b are easily determined; which being found we readily find a' and b' from

$$\left.\begin{aligned} \frac{a'}{a'^2 + b'^2}(a\,b - 2\,C^2) = \frac{C^2}{D} + b \\ \frac{b'}{a'^2 + b'^2}(a\,b - 2\,C^2) = \frac{C^2}{E} + a \end{aligned}\right\} \ldots\ldots (E).$$

Also n will be found from its value in

$$n^2 = \frac{(a^2 + b^2)(a'^2 + b'^2)}{a^2 b^2 - 2\,a\,b\,C^2}.$$

Thus we have not only shown that every equation of two dimensions is that of a conic section, but have also derived the five equations from which may immediately be found the common ordinates of the directrix, the coordinates of the focus, and the constant ratio given by the definition.

211. Ex.—*To find the directrix, the focus, and constant ratio in the conic section*

$$y^2 - 2\,x\,y + x^2 - 8\,x + 16 = 0.$$

First, arranging it homogeneously, we have

$$\frac{x^2}{16} + \frac{y^2}{16} + \frac{x\,y}{-8} + \frac{x}{-2} + 1;$$

$\therefore\ A = 4 = B,\ C^2 = -8,\ C = -2,\ E = \infty;$

$\therefore$ the five equations become

$$\left.\begin{aligned} a^2 - b^2 = 0 \\ a^2 + (b + 4)^2 = (a\,b + 16)\left(\frac{b}{a} + 1\right) \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{a'}{a'^2 + b'^2} \cdot (a b + 16) &= 4 + b \\ \frac{b'}{a'^2 + b'^2} (a b + 16) &= a \end{aligned}\right\}$$

$$n^2 = \frac{(a^2 + b^2)(a'^2 + b'^2)}{a^2 b^2 + 16 a b}.$$

From the first two $a = b = 2$;

$$\left.\begin{aligned} \therefore \quad \frac{a'}{a'^2 + b'^2} &= \frac{3}{10} \\ \frac{b'}{a'^2 + b'^2} &= \frac{1}{10} \end{aligned}\right\} \begin{aligned} &\therefore \quad \frac{1}{a'^2 + b'^2} = \frac{9 + 1}{100} = \frac{1}{10} \\ &\therefore \quad a' = 3, \; b' = 1. \end{aligned}$$

Hence $$n^2 = \frac{8 \times 10}{16 + 64} = 1;$$

$\therefore$ $$n = 1,$$

and the curve is a parabola, whose focus is (3, 1) and directrix $$\frac{x}{2} + \frac{y}{2} = 1.$$

In all cases, by solving the above five equations, the focus, directrix, and constant ratio being determined, the conic section belonging to any equation of two dimensions will be found and fully determined. But instead of having to commit these equations to memory, it is better in practice to simplify the equation by changing the position of the co-ordinates. This shall form the subject of the next section.

SECTION VIII.

TRANSPOSITION OF CO-ORDINATES.

212. PROP.—*To express the co-ordinates* (x, y) *of a point in one system of rectangular co-ordinates in terms of those* (x', y') *in another system of rectangular co-ordinates, of which the axes of* x *and* y *are the right lines*

$$\frac{x}{a} + \frac{y}{b} = 1, \; \frac{x}{a'} + \frac{y}{b'} = 1,$$

respectively. (Fig. 55.)

(First, it must be observed that because these right lines are at right angles $\frac{b}{a} = -\frac{a'}{b'}$. For symmetry, however, we shall retain all of a, b, a', b'.)

Since x' is $\perp$ to the right line $\frac{x}{a'} + \frac{y}{b'} = 1$, and passes through (x, y), $\therefore$ (Art. 51.)

$$x' = \left(\frac{x}{a'} + \frac{y}{b'} - 1\right) \cdot \frac{a' b'}{\sqrt{(a'^2 + b'^2)}};$$

$$\therefore \qquad \frac{x}{a'} + \frac{y}{b'} = 1 + \frac{\sqrt{(a'^2 + b'^2)}}{a' b'} x'.$$

Similarly, $y' = \left(\frac{x}{a} + \frac{y}{b} - 1\right) \frac{a b}{\sqrt{(a^2 + b^2)}}$,

and $$\frac{x}{a} + \frac{y}{b} = 1 + \frac{\sqrt{(a^2 + b^2)}}{a b} x.$$

OTHERWISE.

Let OX, OY be the first axes, OX′, OY′ the new axes; then if y cut the new axes in Q and Q′ we have

$$\left.\begin{aligned}\frac{x}{a} + \frac{QM}{b} &= 1\\ \frac{x}{a'} + \frac{Q'M'}{b'} &= 1\end{aligned}\right\},$$

or $\dfrac{x}{a} + \dfrac{y - PQ}{b} = 1,\ \dfrac{x}{a'} + \dfrac{y + PQ'}{b'} = 1;$

$$\therefore\ \frac{x}{a} + \frac{y}{b} = 1 + \frac{PQ}{b},\ \frac{x}{a'} + \frac{y}{b'} = 1 - \frac{PQ'}{b'}.$$

But $\quad PQ = PM'.\dfrac{AB}{OA} = y'.\dfrac{\sqrt{(a^2 + b^2)}}{a}$

$$PQ' = OM'.\frac{A'B'}{OA'} = x'.\frac{\sqrt{(a'^2 + b'^2)}}{a'};$$

$$\therefore\ \left.\begin{aligned}\frac{x}{a} + \frac{y}{b} &= 1 + \frac{y'}{b}.\ \frac{\sqrt{(a^2 + b^2)}}{a}\\ \frac{x}{a'} + \frac{y}{b'} &= 1 - \frac{x'}{a'}.\ \frac{\sqrt{(a'^2 + b'^2)}}{b'}\end{aligned}\right\}$$

From these simple simultaneous equations x and y may easily be found as required.

213. Cor. 1.—When x and x' are parallel; then $a = \infty,\ b' = \infty;$

$\therefore\quad \dfrac{x}{a} = 0$, and the equations become

$$\left.\begin{aligned}\frac{y}{b} &= 1 + \frac{y'}{b}\\ \frac{x}{a'} &= 1 - \frac{x'}{a'}\end{aligned}\right\}\quad \text{or,}\quad \begin{aligned}y &= y' + b\\ x &= x' - a'.\end{aligned}$$

214. Cor. 2.—When x' is $\perp$ to x; then $b = \infty$ and $a' = \infty$,

and
$$\left.\begin{aligned}\frac{x}{a} &= 1 + \frac{y'}{a}\\ \frac{y}{b'} &= 1 - \frac{x'}{b'}\end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned}x &= a + y'\\ y &= b' - x'\end{aligned}\right\}.$$

215. Prop.—*To express the co-ordinates (x, y) of a point in one system of oblique co-ordinates, in terms of the co-ordinates (x', y') of the same point, in another system of oblique co-ordinates, the axes of which are the right lines*

$$\frac{x}{a} + \frac{y}{b} = 1, \; \frac{x}{a'} + \frac{y}{b'} = 1. \quad \text{(Fig. 56.)}$$

Let O′X′, O′Y′ be the new axes, O X, O Y those given; then

$$\frac{x}{a} + \frac{\mathrm{QM}}{b} = 1,$$

or
$$\frac{x}{a} + \frac{\mathrm{PM} - \mathrm{PQ}}{b} = 1;$$

∴
$$\frac{x}{a} + \frac{y}{b} = 1 + \frac{\mathrm{PQ}}{b}$$

$$= 1 + \frac{\mathrm{PM}'}{b} \cdot \frac{\sin.(x'y')}{\sin.(x'y)} = 1 + \frac{y'}{b} \cdot \frac{\sin.(x'y')}{\sin.(x'y)}.$$

Also
$$\frac{x}{a'} + \frac{\mathrm{Q'M}}{b'} = 1;$$

or
$$\frac{x}{a'} + \frac{\mathrm{PM} - \mathrm{PQ'}}{b'} = 1;$$

or
$$\frac{x}{a'} + \frac{y}{b'} = 1 + \frac{\mathrm{PQ'}}{b'} = 1 + \frac{x'}{b'} \cdot \frac{\sin.(x'y')}{\sin.(y'y)};$$

∴ the simultaneous equations from which x and y are to be obtained are

$$\left.\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 1 + \frac{y'}{b} \cdot \frac{\sin.\,(x'\,y')}{\sin.\,(x'\,y)} \\ \frac{x}{a'} + \frac{y}{b'} &= 1 + \frac{x'}{b'} \cdot \frac{\sin.\,(x'\,y')}{\sin.\,(y'\,y)} \end{aligned}\right\}.$$

By aid of these propositions, or rather of the latter alone, we may change the co-ordinate axis of any curve whose equation is given to any other axes whatever.

Example.

To transpose the axes of the ellipse $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$, *to those of which the origin is the vertex and the axis* x *the focal axis.*

Here the new axes being

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \frac{x}{a'} + \frac{y}{b'} = 1,$$

we have

$$a = \infty, b = o$$
$$a' = A, b' = \infty$$

$$\frac{x}{\infty} + \frac{y}{o} = 1 + y' \sqrt{\left(\frac{1}{o^2} + \frac{1}{\infty^2}\right)},$$

or $$y = y';$$

also $$\frac{x}{A} = 1 - \frac{x'}{A} \cdot \sqrt{\left(\frac{a'^2}{\infty} + 1\right)}$$
$$= 1 - \frac{x'}{A};$$

$$x = A - x';$$

$$\therefore \quad \frac{(A - x')^2}{A^2} + \frac{y'^2}{B^2} = 1;$$

$$\therefore \qquad \frac{x'^2}{A^2} - \frac{y'^2}{B^2} = 2\,\frac{x'}{A},$$

as it ought to be.

216. Prop.—*To simplify the equation of any curve by a transposition of co-ordinates.*

Since

$$\left.\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 1 + \frac{y'}{b} \cdot \frac{\sqrt{(a^2 + b^2)}}{a} \\ \frac{x}{a'} + \frac{y}{b'} &= 1 - \frac{x'}{a'} \cdot \frac{\sqrt{(a'^2 + b'^2)}}{b'} \end{aligned}\right\},$$

in which $\frac{x}{a} + \frac{y}{b} = 1, \frac{x}{a'} + \frac{y}{b'} = 1$ are the equations of the new rectangular axes, and $\therefore \frac{b'}{a'} = -\frac{a}{b}$; these being resolved, and b' eliminated, we get

$$\left.\begin{aligned} x &= \frac{ab^2 + a^2 a'}{a^2 + b^2} + \frac{a}{\sqrt{(a^2 + b^2)}}\,x' + \frac{b}{\sqrt{(a^2 + b^2)}}\,y' \\ y &= \frac{(a - a')\,ab}{a^2 + b^2} - \frac{b}{\sqrt{(a^2 + b^2)}}\,x' + \frac{a}{\sqrt{(a^2 + b^2)}}\,y' \end{aligned}\right\}.$$

But $\frac{a}{\sqrt{(a^2 + b^2)}} = \cos.\,(x'x) = \cos.\,\theta$ suppose,

and $\frac{b}{\sqrt{(a^2 + b^2)}} = \sin.\,(x'x) = \sin.\,\theta.$

Hence we may assume

$$\begin{aligned} x &= A + x' \cos.\,\theta + y' \sin.\,\theta \\ y &= B + x' \sin.\,\theta + y' \cos.\,\theta\,; \end{aligned}$$

and these being substituted in the given equation of the curve, we may destroy three of the terms of the resulting

equation by assuming the co-efficients of such terms separately equal to zero, and thence determining A, B, and θ.

217. Ex. 1.—*To reduce the general equation of two dimensions, viz.*—

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{x\,y}{c^2} + \frac{x}{d} + \frac{y}{e} + 1 = 0.$$

Substituting for x and y and arranging the terms, we get

$$0 = \left(\frac{\cos.^2\theta}{a^2} + \frac{\sin.^2\theta}{b^2} + \frac{\sin.\theta\cos.\theta}{c^2}\right)x'^2$$

$$+ \left(\frac{\cos.^2\theta}{b^2} + \frac{\sin.^2\theta}{a^2} + \frac{\sin.\theta\cos.\theta}{c^2}\right)y'^2$$

$$+ \left(\frac{\sin.2\theta}{a^2} + \frac{\sin.2\theta}{b^2} + \frac{1}{c^2}\right)x'y'$$

$$+ \left(\frac{2\,\mathrm{A}\cos.\theta}{a^2} + \frac{2\,\mathrm{B}\sin.\theta}{b^2} + \frac{\mathrm{A}\sin.\theta + \mathrm{B}\cos.\theta}{c^2}\right.$$

$$\left. + \frac{\cos.\theta}{d} + \frac{\sin.\theta}{e}\right)x' + \left(\frac{2\,\mathrm{A}\sin.\theta}{a^2} + \frac{2\,\mathrm{B}\cos.\theta}{b^2}\right.$$

$$\left. + \frac{\mathrm{A}\cos.\theta + \mathrm{B}\sin.\theta}{c^2} + \frac{\sin.\theta}{d} + \frac{\cos.\theta}{e}\right)y'$$

$$+ \frac{\mathrm{A}^2}{a^2} + \frac{\mathrm{B}^2}{b^2} + \frac{\mathrm{A\,B}}{c^2} + \frac{\mathrm{A}}{d} + \frac{\mathrm{B}}{e} + 1.$$

Now, to reduce this equation, we may destroy any one of its three first terms by assuming its co-efficient $= 0$, and these being independent, A and B containing only θ, we cannot destroy more than one of them. Also, since the three last terms contain A, B, and θ, it hence appears that we may destroy any one of the first three,

and any two of the latter three terms. For instance, assuming

$$\left.\begin{aligned}
&\sin. 2\theta \left(\frac{1}{a^2} + \frac{1}{b^2}\right) + \frac{1}{c^2} = 0 \\
&\left(\frac{2A}{a^2} + \frac{B}{c^2} + \frac{1}{d}\right) \cos. \theta \\
&+ \left(\frac{2B}{b^2} + \frac{A}{c^2} + \frac{1}{e}\right) \sin. \theta = 0 \\
&\left(\frac{2A}{a^2} + \frac{B}{c^2} + \frac{1}{d}\right) \sin. \theta \\
&+ \left(\frac{2B}{b^2} + \frac{A}{c^2} + \frac{1}{e}\right) \cos. \theta = 0
\end{aligned}\right\}$$

$$\sin. 2\theta = -\frac{a^2 b^2}{c^2} \cdot \frac{1}{a^2 + b^2}.$$

$$\left(\frac{2A}{a^2} + \frac{B}{c^2} + \frac{1}{d}\right) \sin. 2\theta$$

$$+ 2\left(\frac{2B}{b^2} + \frac{A}{c^2} + \frac{1}{e}\right) \sin.^2 \theta = 0$$

$$\left(\frac{2A}{a^2} + \frac{B}{c^2} + \frac{1}{d}\right) \sin. 2\theta$$

$$+ 2\left(\frac{2B}{b^2} + \frac{A}{c^2} + \frac{1}{e}\right) \cos.^2 \theta = 0$$

$$\therefore -\frac{a^2 b^2}{c^2} \cdot \frac{1}{a^2 + b^2} \cdot \left(\frac{2A}{a^2} + \frac{B}{c^2} + \frac{1}{d}\right)$$

$$+ 2\left(\frac{2B}{b^2} + \frac{A}{c^2} + \frac{1}{e}\right) = 0$$

Similarly,

$$-\frac{a^2 b^2}{c^2} \cdot \frac{1}{a^2 + b^2}\left(\frac{2B}{b^2} + \frac{A}{c^2} + \frac{1}{e}\right)$$

$$+ 2\left(\frac{2A}{a^2} + \frac{B}{c^2} + \frac{1}{e}\right) = 0;$$

$$\left.\begin{array}{l} \therefore\ \dfrac{2A}{a^2} + \dfrac{B}{c^2} + \dfrac{1}{d} = 0 \\ \text{and}\ \dfrac{2B}{b^2} + \dfrac{A}{c^2} + \dfrac{1}{e} = 0 \end{array}\right\};$$

from which A and B being found, the equation will be reduced very easily to the form

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1;$$

which form shows that the origin of co-ordinates is the centre of the conic section, and the axis of x the focal axis, and a', b' the semi-focal axes.

Similarly, the general equation may be reduced to the simplest general form of a conic section, viz.—

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 2 \cdot \frac{x}{a'}.$$

Particular Instances.

Ex. 2.—*To reduce* $y^2 - 2xy + 3x^2 - 2y - 4x + 5 = 0$ *to the form* $\frac{x^2}{a^2} + \frac{y^2}{b'^2} = 2\frac{x}{a}$.

$$\text{Let } x = A + x' \cos.\theta + y' \sin.\theta$$
$$y = B + x' \sin.\theta + y' \cos.\theta.$$

$\therefore$ substituting, we get

$$(3\cos.^2\theta + \sin.^2\theta - 2\sin.\theta\cos.\theta)\,x'^2$$
$$+ (3\sin.^2\theta + \cos.^2\theta - 2\sin.\theta \,.\, \cos.\theta)\,y'^2$$
$$+ (6\sin.\theta\cos.\theta + 2\sin.\theta\cos.\theta - 2)\,x'y'$$
$$+ (6A\cos.\theta + 2B\sin.\theta - 2\sin.\theta$$

$$- 2\,A \sin.\theta - 2\,B \cos.\theta - 4 \cos.\theta)\,x'$$
$$+ (6\,A \sin.\theta + 2\,B \cos.\theta$$
$$- 2\,A \cos.\theta - 2\,B \sin.\theta - 4 \sin.\theta - 2 \cos.\theta)\,y'$$
$$+ 3\,A^2 + B^2 - 2\,A\,B - 4\,A - 2\,B + 5 = 0.$$

Assume $0 = 8 \sin.\theta.\cos.\theta - 2$

$$\begin{cases} 0 = (6\,A - 2\,B - 4)\sin.\theta + (2\,B - 2\,A - 2)\cos.\theta \\ 0 = 3\,A^2 + B^2 - 2\,A\,B - 4\,A - 2\,B + 5; \end{cases}$$

$$\therefore \sin.\theta\,.\cos.\theta = \frac{1}{4} \text{ and } \sin.2\theta = \frac{1}{2};$$

$$\therefore \sin.\theta = \frac{1}{4}(\sqrt{6} \pm \sqrt{2});\ \cos.\theta = \frac{1}{4}(\sqrt{6} \mp \sqrt{2});$$

whence A and B may be found from the other two equations.

Ex. 3.— *To reduce* $y^2 - x\,y + \frac{x^2}{2} - x + \frac{1}{2} = 0$ *to the simplest general form of the conic sections, viz.,* $\frac{x}{a^2} + \frac{y^2}{b^2} = 2\,.\,\frac{x}{a}$, *and to determine its nature,*

Assume

$$x = A + x' \cos.\theta + y' \sin.\theta$$
$$y = B + x' \sin.\theta + x' \cos.\theta,$$

and substituting, we get

$$0 = \left(\frac{\cos.^2\theta}{2} + \sin.^2\theta - 2\sin.\theta\cos.\theta\right)x'^2$$
$$+ \left(\frac{\sin.^2\theta}{2} + \cos.^2\theta - 2\sin.\theta\cos.\theta\right)y'^2$$
$$+ (\sin.\theta\cos.\theta + 2\sin.\theta\cos.\theta - \sin.\theta\cos.\theta$$
$$- \sin.\theta\cos.\theta)\,x'\,y'$$

$$+ \left(A \cos.\theta + 2B \sin.\theta - \frac{\cos.\theta}{2}\right) x'$$

$$+ (A \sin.\theta + 2B \cos.\theta) y'$$

$$+ \frac{A^2}{2} + B^2 - AB - A + \frac{1}{2}$$

Assume $\therefore$

$$\begin{cases} 0 = \sin.\theta \,.\, \cos.\theta \\ 0 = A \sin.\theta + 2B \cos.\theta \\ 0 = \frac{A^2}{2} + B^2 - AB - A + \frac{1}{2} \end{cases}$$

$\therefore$ $\sin.\theta = 0$, or $\cos.\theta = 0$;

$\therefore$ $B = 0$, or $A = 0$,

that is, $A^2 - 2A + 1 = 0$,

or, $B^2 + \frac{1}{2} = 0$;

this latter, however, cannot obtain ; for then B is imaginary.

$\therefore$ $A^2 - 2A + 1 = 0$, $B = 0$ and $\sin.\theta = 0$;

$\therefore$ the equation becomes

$$0 = \frac{x'^2}{2} + y'^2 + \frac{x'}{2}$$

or, $$\frac{x'^2}{\left(\frac{1}{4}\right)} + \frac{y'^2}{\left(\frac{1}{8}\right)} = 2 \,.\, \frac{x'}{\left(-\frac{1}{2}\right)};$$

$\therefore$ the conic section is an ellipse whose semi-axes are $-\frac{1}{2}$ and $\frac{1}{\sqrt{8}}$.

Ex. 4.—*To reduce* $3x^2 + 4y^2 + 4xy + 3x + 12y$

$- 12 = 0$ *to the form* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 \cdot \frac{x}{a}$ *and to determine the nature of the conic sections.*

Assume $x = A + x' \cos.\theta + y' \sin.\theta$

$y = B + x' \sin.\theta + y' \cos.\theta$;

then,

$$\begin{aligned} 0 = &(3\cos.^2\theta + 4\sin.^2\theta + 4\sin.\theta.\cos.\theta)\, x'^2 \\ &+ (3\sin.^2\theta + 4\cos.^2\theta + 4\sin.\theta\cos.\theta)\, y'^2 \\ &+ (6\sin.\theta\cos.\theta + 8\sin.\theta\cos.\theta + 4)\, x'y' \\ &+ (6A\cos.\theta + 8B\sin.\theta + 4B\cos.\theta \\ &+ 4A\sin.\theta + 3\cos.\theta + 12\sin.\theta)\, x' \\ &+ (6A\sin.\theta + 8A\cos.\theta + 4A\cos.\theta \\ &+ 4B\sin.\theta + 3\sin.\theta + 12\cos.\theta)\, y' \\ &+ 3A^2 + 4B^2 + 4AB + 3A + 4B - 12; \end{aligned}$$

$\therefore$ assuming

$$\begin{cases} 0 = 7\sin.\theta\cos.\theta + 2 \\ 0 = (6A + 4B + 3)\sin.\theta + (4A + 8B + 12)\cos.\theta \\ 0 = 3A^2 + 4B^2 + 4AB + 3A + 4B - 12, \end{cases}$$

A, B and θ may be found as before, which being substituted in

$$\begin{aligned} 0 = &(3\cos.^2\theta + 4\sin.^2\theta + 4\sin.\theta\cos.\theta)\, x'^2 \\ &+ (3\sin.^2\theta + 4\cos.^2\theta + 4\sin.\theta\cos.\theta)\, y'^2 \\ + &\{(6A + 4B + 3)\cos.\theta + (4A + 8B + 12)\sin.\theta\}\, x' \end{aligned}$$

will give the form required.

SECTION IX.

DISCUSSION OF THE COMPLETE EQUATION OF TWO DIMENSIONS.

218. DEF.—A DIAMETER OF A CURVE is any right line which bisects all the chords of the curve that can be drawn from every point of it parallel to one another.

219. DEF.—The CENTRE OF A CURVE is the common point of all its diameters.

220. PROP.—*To find the conditions that the complete equation of two dimensions, viz.*

$$A x^2 + B y^2 + C x y + D x + E y + F = 0$$

may belong to any particular conic sections.

Solving the quadratic, we get

$$x + \frac{C y + D}{2 A} = \pm \frac{1}{2 A} \sqrt{\{(C^2 - 4 A B) y^2 + 2 (C D - 2 A E) y + D^2 - 4 A F\}}.$$

Also,

$$y + \frac{C x + E}{2 B} = \pm \frac{1}{2 B} \sqrt{\{(C^2 - 4 A B) x^2 + 2 (C E - 2 B D) x + E^2 - 4 B F\}}.$$

Hence it appears that

(1.) The equation belongs to two right lines, when

$$C^2 - 4 A B = 0 \text{ and } C D - 2 A E = 0;$$

that is, when $C E - 2 B D = 0$;

also when $C^2 - 4 A B = 0, C D - 2 A E = 0$

and $D^2 - 4 A F = 0$.

(2.) That the equation belongs to a circle when

$$C = 0 \text{ and } A = B; \text{ for then}$$

$$\left(x + \frac{D}{2A}\right)^2 = -y^2 - \frac{E}{A}y + \frac{D^2 - 4AF}{4A^2};$$

or,

$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2A}\right)^2 = \frac{D^2 + E^2 - 4AF}{4A^2},$$

the equation of a circle whose centre is $\left(-\frac{D}{2A}, -\frac{E}{2A}\right)$

and radius $= \sqrt{\frac{D^2 + E^2 - 4AF}{4A^2}}$.

(3.) That the equation belongs to a parabola, when

$$C^2 - 4AB = 0;$$

$$\therefore \left(x + \frac{Cy}{2A}\right)^2 + \frac{D}{A}x + \left(\frac{C}{A} - \frac{CD - 2AE}{2A^2}\right)y + \frac{F}{A} = 0,$$

which is the equation of a parabola.

(4.) That the equation belongs to an ellipse, when

$$C^2 - 4AB$$

is negative; for then x and y are limited, because if y were taken beyond a certain magnitude, x would become imaginary.

(5.) That the equation belongs to a hyperbola when

$$C^2 - 4AB$$

is positive; for then the curve has four infinite branches.

221. Prop.—*To trace the conic section in each of these five cases.*

(1.) When $C^2 - 4AB = 0$ and $CD - 2AE = 0$, then the general equation becomes

$$x + \frac{Cy + D}{2A} = \pm \frac{1}{2A}\sqrt{(D^2 - 4AF)};$$

$$\therefore \frac{x}{\dfrac{D \pm \sqrt{(D^2 - 4AF)}}{2A}} + \frac{y}{\dfrac{D \pm \sqrt{(D^2 - 4AF)}}{C}} = 1;$$

$\therefore$ taking points in the axes of x and y distant from the origin by $\dfrac{D + \sqrt{(D^2 - 4AF)}}{2A}$ and $\dfrac{D + \sqrt{(D^2 - 4AF)}}{C}$, respectively the right line which joins them is that required, and similarly the other right line may be described.

Since $\dfrac{D + \sqrt{(D^2 - 4AF)}}{2A} \div \dfrac{D + \sqrt{(D^2 - 4AF)}}{C}$

$$= \frac{D - \sqrt{(D^2 - 4AF)}}{2A} \div \frac{D - \sqrt{(D^2 - 4AF)}}{C}$$

these right lines are parallel.

When also $D^2 - 4AF = 0$, the right line is

$$\frac{x}{\left(-\dfrac{D}{2A}\right)} + \frac{y}{\left(-\dfrac{D}{2AC}\right)} = 1;$$

two of whose points are

$$\left(-\frac{D}{2A}, 0\right), \left(0, -\frac{D}{2AC}\right).$$

222.—(2.) When $C = 0$, and $A = B$, the equation becomes

$$\left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2A}\right)^2 = \frac{D^2 + E^2 - 4AF}{4A2};$$

$\therefore$ the centre of the circle is $\left(-\frac{D}{2A}, -\frac{E}{2A}\right)$ and the radius $= \sqrt{\frac{D^2 + E^2 - 4AF}{4A^2}}$;

$\therefore$ the circle is easily described.

223. (3.)—When $C^2 - 4AB = 0$ the equation becomes

$$x = -\frac{Cy + D}{2A}$$

$$\pm \frac{1}{2A} \sqrt{\{2(CD - 2AE)y + D^2 - 4AF\}}.$$

First $x = -\frac{Cy + D}{2A}$ or $\frac{x}{\left(-\frac{D}{2A}\right)} + \frac{y}{\left(-\frac{D}{2AC}\right)} = 1$,

is the equation of a diameter of the parabola; for it evidently bisects all the chords that are parallel to the axis of y. Similarly

$$y = -\frac{Cx + E}{2B} \text{ or } \frac{x}{\left(-\frac{E}{C}\right)} + -\frac{y}{\left(-\frac{E}{2B}\right)} = 1.$$

is that diameter which bisects all chords parallel to the axis of x.

Let $2(CD - 2AE)y + D^2 - 4AF = 0$

$$\therefore y = -\frac{D^2 - 4AF}{2(CD - 2AE)},$$

$$\therefore x = \frac{4A(ED - CF) - CD^2}{4A(CD - 2AE)},$$

which are the co-ordinates of a point in the curve at which y is the tangent. This is evidently the least value of x.

Similarly, by assuming

$$2(CE-2BD)+E^2-4BF=0,$$

we shall find the least value of y and the point at which x becomes a tangent.

$$\text{Let } x=\infty\ ;\ \text{then } y=\pm\infty,$$

or the curve has two infinite branches.

As many points of the curve may be found as are required to trace its figure, by assuming values for one co-ordinate and finding the corresponding values of the other from the equation.

224. (4.)—When C^2-4AB is negative, let

$$C^2-4AB=-M,\ 2(CD-2AE)=N,$$
$$2(CE-2BD)=N',$$
$$D^2-4AF=P.$$

then
$$E^2-4BF=P'$$

$$x+\frac{Cy+D}{2A}=\pm\frac{1}{2A}\sqrt{(-My^2+Ny+P)},$$

$$y+\frac{Cx+E}{2B}=\pm\frac{1}{2B}\sqrt{(-Mx^2+N'x+P)}.$$

As before

$$x+\frac{Cy+D}{2A}=0,\ y+\frac{Cx+E}{2B}=0,$$

$$\text{or, } \frac{x}{\left(-\frac{D}{2A}\right)}+\frac{y}{\left(-\frac{D}{2AC}\right)}=1,$$

$$\text{and } \frac{x}{\left(-\frac{E}{C}\right)}+\frac{y}{\left(\frac{E}{2B}\right)}=1,$$

are the equations of the diameters that bisect all chords parallel to the axes of y and x respectively.

Also, neither y nor x can exceed a certain quantity, for

$$-My^2 + Ny + P \text{ cannot be} < 0,$$
$$-Mx^2 + N'x + P' \text{ cannot be} < 0.$$

Let $\quad -My^2 + Ny + P = 0,$

or $$y^2 - \frac{N}{M}y = \frac{P}{M},$$

$$\therefore \quad y = \frac{N \pm \sqrt{(N^2 + 4MP)}}{2M};$$

which are the maximum and minimum values of y. Similarly, the maximum and minimum values of x are

$$\frac{N' \pm \sqrt{(N'^2 + 4MP')}}{2M}.$$

Hence

$$x = -\frac{C}{2A}y - \frac{D}{2A} =$$
$$-\frac{C}{4AM}\cdot\{N \pm \sqrt{(N^2 + 4MP)}\} - \frac{D}{2A}$$

and y as before, will give the two points at which y is a tangent to the ellipse

$$\text{and} \quad y = -\frac{C}{2B}x - \frac{E}{2B} =$$
$$-\frac{C}{4BM}\cdot\{N' \pm \sqrt{(N'^2 + 4MP')}\} - \frac{E}{2B},$$
$$\text{and } x = \frac{N' \pm \sqrt{(N'^2 + 4MP')}}{2M},$$

will give the two points at which x is a tangent to the ellipse.

Any number of points being found in the ellipse, it may be traced between these tangents or limits.

225. (5.)—When $C^2 - 4AB$ is positive; then the equation becomes by the same assumptions

$$x + \frac{Cy + D}{2A} = \pm \frac{1}{2A}\sqrt{(My^2 + Ny + P)},$$

$$y + \frac{Cx + E}{2B} = \pm \frac{1}{2B}\sqrt{(Mx^2 + N'x + P')}.$$

As before

$$\left.\begin{array}{l} \dfrac{x}{\left(-\dfrac{D}{2A}\right)} + \dfrac{y}{\left(-\dfrac{D}{2AC}\right)} = 1 \\[2ex] \dfrac{x}{\left(-\dfrac{E}{C}\right)} + \dfrac{y}{\left(\dfrac{E}{2B}\right)} = 1 \end{array}\right\},$$

are the equations of the diameters that bisect all chords parallel to the axes of y and x respectively.

Let $x = \pm \infty$; then $y = \pm \infty$,

or the hyperbola has four infinite branches.

Also $My^2 + Ny + P$, or $Mx^2 + N'x + P'$ cannot be < 0 or negative; that is, their least values are zero; hence the maximum and minimum values of x and y are derived from

$$My^2 + Ny + P = 0, \ Mx^2 + N'x + P' = 0;$$

from which

$$y + \tfrac{1}{2}\frac{N}{M} = \pm \sqrt{\left(\frac{N^2}{4M^2} - P\right)},$$

$$x + \tfrac{1}{2}\frac{N'}{M} = \pm \sqrt{\left(\frac{N'^2}{4M^2} - P'\right)},$$

which two values of y and x being substituted in

$$x + \frac{Cy + D}{2A} = 0\,,\; y + \frac{Cx + E}{2B} = 0,$$

will give the corresponding values of x and y, and thence we obtain the points at which the co-ordinates of y and x touch the hyperbola respectively, thus determining the limits of the curve.

Also $\quad x + \dfrac{Cy + D}{2A} = \pm \dfrac{y\sqrt{M}}{2A}\left(1 + \dfrac{Ny + P}{My^2}\right)^{\frac{1}{2}}$

$$= \pm \frac{y\sqrt{M}}{2A}\left(1 + \tfrac{1}{2}\frac{Ny + P}{My^2} - \tfrac{1}{8}\cdot\frac{(Ny+P)^2}{M^2y^4} - \&c.\right),$$

$$= \pm \frac{y\sqrt{M}}{2A} \pm \frac{N}{4A\sqrt{M}} \pm \frac{Q}{y} + \frac{R}{y^2} + \&c.,$$

$\pm$ Q and R being the co-efficients of $\dfrac{1}{y}$, $\dfrac{1}{y^2}$, &c.,

$$\therefore\; x + \frac{Cy + D}{2A} = \pm \frac{y\sqrt{M}}{2A} \pm \frac{N}{4A\sqrt{M}},$$

or $\quad x + \dfrac{C \mp \sqrt{M}}{2A}y = -\dfrac{D}{2A} \pm \dfrac{N}{4A\sqrt{M}};$

$$\therefore\quad \frac{x}{-\dfrac{D}{2A} \pm \dfrac{N}{4A\sqrt{M}}} + \frac{y}{\dfrac{-2D\sqrt{M} \pm N}{2(C \mp \sqrt{M})\sqrt{M}}} = 1,$$

which are the equations of the asymptotes of the hyperbola.

Any number of points in the hyperbola may be found by assuming values of one co-ordinate and finding the other corresponding co-ordinates from the equation.

Section X.

SECTIONS OF THE CONE BY A PLANE.

226. Def.—A Cone *is a surface, generated by a right line passing through a given point and through every point of a circle not in the same plane with that point.* (Fig. 57.)

Thus, if a be the given point and $b\,c\,d$ the given circle, the surface generated by the right line $a\,b$ passing through every point of the circle $b\,e\,d$ is that of a cone.

227. Def.—A Right Cone is that in which the right line joining the given point and the centre of the circle is at right angles to the plane of that circle.

228. Def.—An Oblique Cone is that in which the right line joining the given point, and the centre of the circle is not at right angles to the plane of that circle.

229. Def.—The Axis of a Cone is the right line joining the given point and the centre of the circle.

230. Def.—The Base of the Cone is the generating circle.

231. Def.—The Vertex of the Cone is the given point.

232. Def.—The Sheets of a Cone are the surfaces which are separated by the vertex.

233. Prop.—*To find the equation of the section of a right cone made by a plane, the axes of co-ordinates being in that plane and origin the intersection of the generating right line with the cutting plane, when the*

cutting plane and plane passing through the centre and generating line are at right angles to one another.

Let the plane Y A X cut the cone $a\,b\,d$ making the section A P A′, the plane Y A X being $\perp$ plane $a\,b\,c$ in which c is the centre, and a the vertex of the cone. Also, let b' P d' be the section made by plane parallel to the base $b\,e\,d$ and intersecting the former plane in the right line P M, and the plane $a\,b\,c$ in $b'\,d'$. Then since the planes A P A′, b P d' are each $\perp$ the plane $a\,b\,c$, $\therefore$ P M is $\perp$ A A′ and also $\perp$ $b'\,d'$. Hence, by nature of the circle

$$\mathrm{P\,M}^2 = b'\,\mathrm{M} \times \mathrm{M}\,d'$$

and calling $\mathrm{A\,M} = x$, $\mathrm{P\,M} = y$, the angle $a\,\mathrm{A\,X} = \alpha$, $\mathrm{A\,A'} = 2\,a$, $\mathrm{A}\,a = \delta$ and $\angle\, b\,a\,d = 2\,\beta$, we get

$$b'\,\mathrm{M} : x :: \sin.\alpha : \cos.\beta\,.$$

Also $\mathrm{M}\,d' : 2\,a - x :: \sin.\,(2\beta + \alpha) : \cos.\,\beta$,

$$\therefore \quad y^2 = x\,.\,\frac{\sin.\,\alpha}{\cos.\,\beta} \times (2a - x)\,.\,\frac{\sin.\,(\alpha + 2\beta)}{\cos.\,\beta}$$

$$= \frac{\sin.\,\alpha\,.\,\sin.\,(\alpha + 2\beta)}{\cos.^2\beta}\,.\,(2\,a\,x - x^2)\,;$$

also $\qquad \delta : 2\,a :: \sin.\,(\alpha + 2\,\beta) : \sin.\,2\,\beta\,;$

$$\therefore\; y^2 = \frac{\sin.\,\alpha\,.\,\sin.\,(\alpha + 2\,\beta)}{\cos.^2\beta}\left(\frac{\delta\,.\,\sin.\,2\,\beta}{\sin.\,(\alpha + 2\,\beta)}\,x - x^2\right),$$

or

$$\frac{x^2}{\left\{\dfrac{\delta\sin.\,2\,\beta}{2\sin.\,(\alpha+2\,\beta)}\right\}^2} + \frac{y^2}{\left(\dfrac{\delta^2\sin.\,\alpha\,.\,\sin.^2\beta}{\sin.\,(\alpha + 2\,\beta)}\right)} = 2\,.\,\frac{x}{\dfrac{\delta\sin.\,2\,\beta}{2\sin.\,(\alpha+2\beta)}}$$

which is the equation of a conic section whose semi-focal and semi non-focal axes are respectively

$$\frac{\delta \sin 2\beta}{2\sin.(\alpha+2\beta)} \text{ and } \sqrt{\frac{\delta^2 \sin.\alpha \sin.^2\beta}{\sin.(\alpha+2\beta)}},$$

and the conic section is a right line, a circle, a parabola, an ellipse, or hyperbola, according as

$\delta \sin.\beta \sqrt{\frac{\sin.\alpha}{\sin.(\alpha+2\beta)}}$ is 0, whilst $\frac{\delta\sin.2\beta}{2\sin.(\alpha+2\beta)}$; is finite; equal to $\frac{\delta\sin.2\beta}{2\sin.(\alpha+2\beta)}$; $\frac{\delta\sin.2\beta}{2\sin.(\alpha+2\beta)} = \infty$; $\delta\sin.\beta\sqrt{\frac{\sin.\alpha}{\sin.(\alpha+2\beta)}}$, unequal to $\frac{\delta\sin.2\beta}{2\sin.(\alpha+2\beta)}$, or imaginary; that is, according as

$\sin.\alpha = 0$, $\alpha = \frac{\pi}{2} - \beta$, $\alpha + 2\beta = \pi$, $\alpha + 2\beta < \pi$ $\alpha + 2\beta > \pi$; and the corresponding equations are

$$y = 0,\ x^2 + y^2 = \delta\sin.\beta.x,\ y^2 = 4\delta\sin.^2\beta.x,$$

$$\frac{x^2}{\left(\frac{\delta\sin.\beta\cos.\beta}{\sin.\gamma}\right)^2} + \frac{y^2}{\left\{\frac{\delta^2\sin.^2\beta.\sin.(2\beta+\gamma)}{\sin.\gamma}\right\}} = 2.\frac{x}{\left(\frac{\delta\sin.\beta\cos.\beta}{\sin.\gamma}\right)}$$

and $$\frac{x^2}{\left(\frac{\delta\sin.\beta\cos.\beta}{\sin.\gamma'}\right)^2} - \frac{y^2}{\left\{\frac{\delta^2\sin.^2\beta\sin.(2\beta-\gamma')}{\sin.\gamma'}\right\}} = -2.\frac{x}{\left(\frac{\delta\sin.\beta\cos.\beta}{\sin.\gamma'}\right)};$$

wherein $\alpha + 2\beta = \pi - \gamma$, $\alpha + 2\beta = \pi + \gamma'$.

In the case of the hyperbola, the cutting plane will cut both sheets of the cone, thus forming the hyperbola and the opposite hyperbola.

Section XI.

OTHER USEFUL CURVES.

The Cycloid.

234. Def.—*A* Cycloid *is a curve generated by a point given in the circumference of a circle, whilst the circle rolls on an indefinite right line in its own plane.* (Fig. 58.)

235. Prop.—*To find the rectangular equation of a cycloid, the axis of* x *being the right line upon which the generating circle rolls and the origin a point of the curve.*

Let P be any point (x, y) of the cycloid, a P b being the corresponding position of the generating circle; A X, A Y the axes. Then since P began to move when at A, P $a\,b$ must $=$ the path A b;

$$\therefore \quad x = b\,\mathrm{M} + \mathrm{A}b = \mathrm{P}m + \mathrm{P}ab$$

$$= \sqrt{(2\,\mathrm{R}\,y - y^2)} + \mathrm{R}\,.\,\text{vers.}^{-1}\,\frac{y}{\mathrm{R}},$$

R being the radius of the generating circle;

$$\text{or} \qquad x = \sqrt{(2\,\mathrm{R}\,y - y^2)} + \mathrm{R}\cos.^{-1}\frac{\mathrm{R}-y}{\mathrm{R}}.$$

1. The base A B of the branch A P B $=$ circumference of generating circle.

2. The greatest value of y is $2R$, whereas x may $= \pm \infty$.

3. From the curve being formed by the momentary revolution of the generating circle about b as a centre, the right line bP is $\perp$ to the tangent at P; that is, the tangent at P coincides with the chord of the generating circle passing through P and its highest point a. Hence it is easily found that the equation of the tangent at the point (a, b) of the cycloid is

$$\frac{x}{a - \dfrac{b\sqrt{b}}{\sqrt{(2R - b)}}} + \frac{y}{b - a\sqrt{\dfrac{2R - b}{b}}} = 1;$$

which may also be easily shown as in the case of the conic sections. Whence the equation to the normal can easily be found.

4. The radius of curvature is

$$\rho = 2\sqrt{(4R^2 - 2Ry)}.$$

5. The evolute of each semi-cycloid is an equal and similar semi-cycloid in an inverted position. The equation of it is

$$x = \sqrt{(2Ry - y^2)} + R \cdot \cos.^{-1}\frac{R + y}{R}.$$

And the diagram represents a series of the branches of a cycloid and of its evolute. (See fig. 59.)

6. The area of one branch of the cycloid is $3\pi R^2$, or three times that of its generating circle.

THE TROCHOID OF NEWTON; OR PROLATE CYCLOID.

236. DEF.—*The* TROCHOID *is a curve described by a given point within a circle during the rolling of the circle along a given right line, and always in its own plane.*

237. Prop.—*To find the equation of a trochoid.* (Fig. 60.)

Let P be any point (x, y) of the trochoid, the circle Q B having rolled along A X from A, and the given point Q in C P produced, having at first been at A.

Draw $Pm \perp$ diameter E D of the concentric circle P E D; then if C Q = R, C P = R′, we have

$$\begin{aligned} x &= \mathrm{AM} = \mathrm{AB} - \mathrm{BM} \\ &= \text{arc } \mathrm{BQ} - \mathrm{P}m \\ &= \text{arc } \mathrm{DP} \times \frac{\mathrm{R}}{\mathrm{R}'} - \sqrt{\{\mathrm{R}'^2 - (y - \mathrm{R})^2\}} \\ &= \mathrm{R} \,.\, \text{vers.}^{-1} \frac{y - (\mathrm{R} - \mathrm{R}')}{\mathrm{R}'} - \sqrt{\{\mathrm{R}'^2 - (y - \mathrm{R})^2\}} \\ &= \mathrm{R} \,.\, \cos.^{-1} \frac{\mathrm{R} - y}{\mathrm{R}'} - \sqrt{\{\mathrm{R}'^2 - (y - \mathrm{R})^2\}}, \end{aligned}$$

the equation required.

This curve Newton uses in the sixth section of the 'Principia' to cut off a sectorial area from an ellipse, proportional to the time that a planet describes that area about the sun in the focus of the ellipse by means of its varying distance from the sun.

The Curtate Cycloid.

238. Def.—*The* Curtate Cycloid *is a curve described by a given point without a circle during the rolling of the circle along a given right line, and always in its own plane.*

239. Prop.—*To find the equation of a curtate cycloid.*

The equation is found, as in the trochoid, to be

$$x = \mathrm{R} \,.\, \cos.^{-1} \frac{\mathrm{R} - y}{\mathrm{R}'} - \sqrt{\{\mathrm{R}'^2 - (y - \mathrm{R})^2\}},$$

the only distinction being; that in the trochoid R is $>$ R′, whereas in the curtate cycloid R′ is $>$ R.

THE EPICYCLOID.

240. DEF.—AN EPICYCLOID *is a curve generated by a given point in the circumference of a circle, whilst it rolls upon the convex circumference of another circle, preserving its plane in the plane of that circle.* (Fig. 61).

The rolling circle is called the *Generating Circle.*

The other circle is the *Path Circle.*

241. PROP.—*To find the equation of the epicycloid.*

Let P be any point of the epicycloid, C being the centre of the path circle, C′ that of the generating circle; and suppose the point P at first to have been at A. Make A C the axis of x, A being the origin, and complete the diagram.

Now the equation must manifestly depend upon the circumstance that the arc P D = arc A D; or that

$$PC' \,.\, \angle\, PC'D = DC \,.\, \angle\, ACD.$$

But, supposing D C = R and P C′ = R′, and calling the co-ordinates of the centre C′, (a, b), we have

$$\cos. ACD = \frac{CN}{CC'} = \frac{R - a}{R + R'};$$

also $$\cos. PC'D = \frac{(R + R')^2 + R'^2 - (x + R)^2 - y^2}{2R'(R + R')};$$

$$\therefore R' \,.\, \cos.^{-1} \frac{(R + R')^2 + R'^2 - (x + R)^2 - y^2}{2R'(R + R')}$$
$$= R \cos.^{-1} \frac{R - a}{R + R'};$$

and it only remains to eliminate a.

But $(x-a)^2+(y-b)^2=R'^2$ and $(R-a)^2+b^2=(R+R')^2$; from which eliminating b we get a; and this being substituted in the above expression will give the equation required.

242.—The polar equation of the epicycloid, referred to the centre of the path-circle as a pole, is

$$\theta + \tan.^{-1}\frac{e}{R}u - \tan.^{-1}\frac{e}{R}\sqrt{\frac{3R+e}{R+3e}} =$$
$$\frac{e}{R}\left(\tan.^{-1}u - \tan.^{-1}\sqrt{\frac{3R+e}{R+3e}}\right);$$

in which $e = R + 2R'$, and $u^2 = \dfrac{r^2 - R^2}{(R+2R')^2 - r^2}$.

243.—If θ be measured from the common point of the epicycloid and path-circle, then

$$\theta = \frac{e}{R}\tan.^{-1}u - \tan.^{-1}\frac{e}{R}u.$$

244.—The equation between the perpendicular (p) on the tangent from the centre of the path-circle and the radius-vector (r) originating in that centre, is

$$p^2 = e^2 \cdot \frac{r^2 - R^2}{e^2 - R^2}.$$

This curve is introduced in Newton's 'Principia,' Section X.

The Hypocycloid.

245. Def.—*A* Hypocycloid *is a curve generated by a given point in the circumference of a circle, whilst it rolls upon the concave circumference of another circle, its plane being always in the plane of that circle.*

246. Prop.—*To find the equation of a Hypocycloid.* (Fig. 62.)

Let P be any point of the hypocycloid, C being the centre of the path-circle, C′ that of the generating circle and suppose the point P, at first, to have been at A. Make A C the axis of x, A being the origin, and complete the diagram. As before

$$\mathrm{PC'}.\angle \mathrm{PC'D} = \mathrm{DC}.\angle \mathrm{ACD},$$

But $\cos. \mathrm{ACD} = \dfrac{\mathrm{CN}}{\mathrm{CC'}} = \dfrac{\mathrm{AC}-\mathrm{AN}}{\mathrm{R}-\mathrm{R'}} = \dfrac{\mathrm{R}-a}{\mathrm{R}-\mathrm{R'}}$;

$$\cos. \mathrm{PC'D} = -\cos. \mathrm{PC'C},$$

$$= -\frac{\mathrm{R'^2} + (\mathrm{R}-\mathrm{R'})^2 - (\mathrm{R}-x)^2 - y^2}{2\mathrm{R'}(\mathrm{R}-\mathrm{R'})};$$

$$\therefore\ \mathrm{R'}.\cos.^{-1} - \frac{(\mathrm{R}-\mathrm{R'})^2 + \mathrm{R'^2} - (\mathrm{R}-x)^2 - y^2}{2\mathrm{R'}(\mathrm{R}-\mathrm{R'})}$$

$$= \mathrm{R}\cos.^{-1}\frac{\mathrm{R}-a}{\mathrm{R}-\mathrm{R'}},$$

from which a being eliminated by means of

$(x-a)^2 + (y-b)^2 = \mathrm{R'^2}$ and $(\mathrm{R}-a)^2 + b^2 = (\mathrm{R}-\mathrm{R'})^2$,

the result will be the equation required.

The Epicycle.

247. Def.—*The* Epicycle *is a curve generated by the uniform motion of a point in the circumference of a circle, the centre of that circle also moving uniformly in the circumference of another fixed circle, and the planes of both circles being coincident.*

248. Prop.—*To find the equation of the* Epicycle. (Fig. 63.)

Let C be the centre of the quiescent circle, C′ that of the revolving circle, and suppose that the moving point P is first at O, the origin of co-ordinates. Then,

when P is at O, if O D the chord be taken $= C'P$, D will be the first situation of the centre C′. Hence, supposing the point P to be the place of the generating point (x, y) C′ P must have revolved through the ∠ A C′ P (C′ A being parallel to O D) whilst C C′ has described the ∠ D C C′. Now, these angular motions are uniform and given.

Let ∴ $\angle ACP' = n. \angle DCC'$; that is, if $\angle DCC'$ be α and $AC'P = \alpha'$,

$$\alpha' = n . \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Also if $CC' = R$, $C'P = R'$ and the co-ordinates of C′ be (a, b);

then

$$R - a = CN = R \cos. OCC' = R . \cos. (\alpha + \angle OCD),$$

$$= R \cos. \left(\alpha + 2 \sin.^{-1} \frac{R'}{2R}\right) \quad . \quad . \quad . \quad . \quad . \quad (2).$$

$$y - b = Pm = R' \sin. \{2\pi - \alpha' - (\pi - DOC)\},$$

$$= R' \sin. \{\pi - (n\alpha - DOC)\}$$

$$= R'. \sin. (n\alpha - DOC),$$

$$= R' \sin. \left(n\alpha - \cos.^{-1} \frac{R'}{2R}\right) . \quad . \quad . \quad . \quad . \quad (2).$$

Also $\quad R^2 = (R - a)^2 + b^2$;

$$\therefore \quad a^2 + b^2 - 2Ra = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

and $\quad R'^2 = (x - a)^2 + (y - b)^2$;

$$\therefore \quad 2Ra - 2x.a - 2y.b = R'^2 - x^2 - y^2;$$

$$\therefore (R - x)a - y.b = \frac{R'^2 - x^2 - y^2}{2} \quad . \quad . \quad . \quad . \quad (4).$$

From these four equations, a, b and α being eliminated, the result will be the equation required.

From (3) and (4), we may find a and b and eliminate

them by substitution in (1) and (2); and from the new forms of equations (1), and (2), a is eliminated. The final result, however, will be very complex.

This is the curve which would be, and which very nearly is, described by the moon or any other satellite, if she or it moved in a circle instead of an ellipse round the earth, and also if the earth moved in a circle round the sun.

An indefinite number of curves may be conceived as generated by a point in one curve, whilst that curve rolls on another. Thus, any given point of an ellipse which rolls on another ellipse would generate a certain curve, and similarly with every species of curve. Curves thus formed are called by French writers *Roulettes*. To this class belong the Epitrochoid and Hypotrochoid which are generated respectively by a point within a circle revolving upon another circle, and by a point without a circle which revolves upon another circle. These curves being rather curious than useful, I merely notice their existence.

Companion of the Cycloid.

249. Def.—*The* Companion of the Cycloid *is a curve generated by a right line moving at right angles to and along the diameter of a circle from its vertex, and always equal in length to the arc intercepted between it and the vertex.*

250. Prop.—*To find the* Equation of the Companion of the Cycloid. (Fig. 64.)

Let P be any point in the curve, the vertex O of the

circle being taken for the origin of co-ordinates and the axis of x the corresponding diameter. Then, by the definition

$$y = \mathrm{P\,M} = \mathrm{O\,P'} = \mathrm{O\,C} \,.\, \angle \mathrm{O\,C\,P'},$$

$$= \mathrm{R} \,.\, \text{vers.}^{-1} \frac{x}{\mathrm{R}},$$

$$\text{or } y = \mathrm{R} \,.\, \cos.^{-1} \frac{\mathrm{R} - x}{\mathrm{R}},$$

the equation required.

THE CATENARY.

251. DEF.—*The* CATENARY *is a curve formed by the action of gravity upon a line, attached to two given points, the line being supposed material and perfectly flexible.*

252. Its equation, as derived by writers on Mechanics, (see *Whewell's Mechanics*) is

$$y = \frac{a}{2} \,.\, \left(e^{\frac{x}{a}} + e^{\frac{-x}{a}} \right) - a \,.$$

in which a is the tension of the string at the lowest point.

THE TRISECTRIX.

253. DEF.—*The* TRISECTRIX *is a curve generated by a point taken in the chord of a circle always passing through the same point of the circumference of the circle, whose distance from the other variable extremity of the chord is equal to the radius of the circle.*

254. PROP.—*To find the equation of the Trisectrix.* (Fig. 65.)

Let P be any point (x, y) of the trisectrix, O Q the corresponding chord of the given circle, and O X the axis of x passing through its centre. Then if R be the radius of that circle, we have

$$\text{R} : \text{ON} - x :: \sqrt{(x^2 + y^2)} : x,$$

$$\text{ON} = x + \frac{\text{R}x}{\sqrt{(x^2 + y^2)}} = x \cdot \frac{\text{R} + \sqrt{(x^2 + y^2)}}{\sqrt{(x^2 + y^2)}}.$$

Also $\quad \text{ON} : \text{OQ} :: \text{OQ} : \text{OA},$

$$\therefore\ \text{ON} = \frac{\text{OQ}^2}{\text{OA}} = \frac{\{\text{R} + \sqrt{(x^2 + y^2)}\}^2}{2\,\text{R}};$$

$$\therefore\ \text{R} + \sqrt{(x^2 + y^2)} = 2\,\text{R} \cdot \frac{x}{\sqrt{(x^2 + y^2)}};$$

$$\therefore\ \text{R}^2 (x^2 + y^2) = (2\,\text{R}x - x^2 - y^2)^2,$$

or $(x^2 + y^2)^2 - (4\,\text{R}x + \text{R}^2)(x^2 + y^2) + 4\,\text{R}^2 x^2 = 0,$

the equation required.

255.—*To find the polar equation of the Trisectrix.*

Let $\angle\, \text{XOP} = \theta$, and $\text{OP} = r$; then

$$r = \text{OQ} \mp \text{PQ}$$

according as P is taken in O Q or O Q produced.

But $\text{OQ} = 2\,\text{R} \cos.\ \theta$, and $\text{PQ} = \text{R}$,

$$\therefore \qquad r = 2\,\text{R} \cos.\ \theta \mp \text{R},$$

the equation required.

The Use of the Trisectrix. (Fig. 66.)

256. *To Trisect an Angle.*

Let the $\angle$ A O B be the angle to be trisected, and O A = O B. Also, let O P B be a branch of the trisectrix, the centre of whose generating circle is B; cutting the chord A B in P; then if it can be shown that $\angle$ A O P = 2 $\angle$ P O B, we have only to bisect the $\angle$ A O P and the $\angle$ A O B will be trisected.

Let $\text{POB} = \theta$, $\text{AOP} = \phi$, $\text{OP} = r$, $\text{OB} = \text{R}$; then, by equation of the trisectrix

$$\text{OP} = r = 2\text{R}.\ \cos.\ \theta - \text{R}.$$

$$\text{But } BP = OP. \frac{\sin. \theta}{\cos. \frac{1}{2}(\theta + \varphi)},$$

$$AP = OP. \frac{\sin. \varphi}{\cos. \frac{1}{2}(\theta + \varphi)};$$

$$\therefore AB = \frac{OP}{\cos. \frac{1}{2}(\theta + \varphi)}(\sin. \theta + \sin. \varphi),$$

$$\text{also} = 2R \,.\, \sin. \frac{1}{2}(\theta + \varphi);$$

$$\therefore OP(\sin. \theta + \sin. \varphi) = R \sin. (\theta + \varphi).$$

$$\text{But } OP = R(2 \cos. \theta - 1);$$

$$\therefore (2 \cos. \theta - 1)(\sin. \theta + \sin. \varphi) = \sin. (\theta + \varphi).$$

$$\text{Hence } \frac{\cos. \theta - \cos. \varphi}{\sin. \theta + \sin. \varphi} = \frac{1 - \cos. \theta}{\sin. \theta};$$

$$\therefore \frac{-\sin. \frac{\theta + \varphi}{2} \sin. \frac{\theta - \varphi}{2}}{\sin. \frac{\theta + \varphi}{2} \cos. \frac{\theta - \varphi}{2}} = \frac{\sin.^2 \frac{\theta}{2}}{\sin. \frac{\theta}{2} \cos. \frac{\theta}{2}}$$

$$\text{or,} \quad -\tan. \frac{\theta - \varphi}{2} = \tan. \frac{\theta}{2};$$

$$\therefore \quad \frac{\varphi - \theta}{2} = \frac{\theta}{2};$$

$$\therefore \quad \varphi = 2\theta.$$

Thus, the trisection of an angle, that famous problem of antiquity, is effected by aid of the trisectrix. It is much more simply done, however, by means of the hyperbola.

This, although somewhat of a digression, we shall here give.

257.—*To trisect any angle by the hyperbola.* (Fig. 67.)

Let B C S be the angle to be trisected, which bisect by the right line C E; draw S F $\perp$ C E and take S A $= 2$ A F; then with S as focus, C E directrix, and $2 : 1$ the constant ratio of the definition, let a branch of a hyperbola A Q be described cutting the arc B S in P. The arc S P $= \dfrac{1}{3}$ S B, and the angle will thence be easily trisected. For S P being joined, P N D drawn $\perp$ C E, and B D joined, S P $=$ 2 P N; and it is easily shown that P D $=$ 2 P N and B D $=$ S P;

$$\therefore \; SP = PD = BD;$$

$\therefore$ the arc and consequently the angle is trisected.

The Quadratrix of Tschirnhausen.

258. Def.—*The* Quadratrix of Tschirnhausen *is the locus of the intersections of two right lines at right angles to one another, one of which moves uniformly from the vertex of a circle to its centre, always at right angles to the radius, whilst the other uniformly passes through the quadrant of that circle.*

259. Prop.—*To find the equation of the quadratrix of Tschirnhausen.* (Fig. 68.)

Let A P′ O be the quadrant, and C its centre; O the origin of the axis of co-ordinates, of which that of x passes through the centre; then if P be any point (x, y) of the quadratrix A P O, the generating right

lines being P P′, P M (the points M and P′ moving along O C, O P′ A so as to pass through them uniformly in the same time), we have

$$y = \mathrm{P\,M} = \mathrm{R}\sin.\ \mathrm{O\,C\,P'};$$

$$\text{But}\quad x : \mathrm{O\,P'} :: \mathrm{O\,C} : \text{arc } \mathrm{O\,A} :: 1 : \frac{\pi}{2};$$

$$\therefore\quad \mathrm{O\,P'} = \frac{\pi}{2}x = \mathrm{R}\,.\,\angle \mathrm{O\,C\,P'}$$

$$\therefore\quad \angle \mathrm{O\,C\,P'} = \frac{\pi}{2}\cdot\frac{x}{\mathrm{R}}$$

$$\therefore\quad y = \mathrm{R}\,.\sin.\left(\frac{\pi}{2}\cdot\frac{x}{\mathrm{R}}\right)$$

is the equation required.

260. The polar equation is

$$r\sin.\ \theta = \mathrm{R}\,.\sin.\left(\frac{\pi}{2}\cdot\frac{r\cos.\ \theta}{\mathrm{R}}\right).$$

It is very easy to trace this curve, which consists of an infinite number of similar and equal branches similarly situated along the axis of x.

Use of the Quadratrix of Tschirnhausen.

261.—*To multisect any given angle.* (Fig. 69.)

Let O C Q be the angle to be multisected; complete the quadrant O Q A and let O P A be a branch of the quadratrix. Draw Q P parallel to O C, meeting the quadratrix in P, P M $\perp$ O C, and take $\mathrm{O}\,m = \frac{1}{n}\mathrm{O\,M}$. Draw $m\,p$ parallel to M P, meeting the quadratrix in p, and $p\,q$ parallel to O C, meeting the circle in q, and

join q C; then $\angle\, q\,\mathrm{C\,O} = \frac{1}{n}\,\angle\,\mathrm{Q\,C\,O}$.

$$\text{For}\quad \mathrm{OQ : OM :: OQA : OC},$$

$$\text{or}\quad \mathrm{R}\,.\,\angle\,\mathrm{OCQ : OM} :: \frac{\pi}{2}\,.\,\mathrm{R : R};$$

$$\therefore\quad \angle\,\mathrm{OCQ} = \frac{\pi}{2}\,.\,\frac{\mathrm{OM}}{\mathrm{R}};$$

$$\text{Similarly,}\quad \angle\,\mathrm{OC}\,q = \frac{\pi}{2}\,.\,\frac{\mathrm{O}\,m}{\mathrm{R}} = \frac{\pi}{2}\,.\,\frac{\mathrm{OM}}{n\,.\,\mathrm{R}};$$

$$\therefore\quad \angle\,\mathrm{OC}\,q = \frac{1}{n}\,.\,\angle\,\mathrm{OCQ}.$$

In the same way the $(n-1)$ th part of $\angle\,\mathrm{QC}\,q$ may be cut off, and so on, until $\angle$ O C Q is cut into n equal parts.

As a particular instance, the quadratrix may be made to supersede the trisectrix or hyperbola in trisecting an angle. Thus, if O M be trisected by the points m, m', the corresponding points q, q' will trisect the arc O Q, &c.

Another use, by far the more important, (for no great value must be attached to the trisection of an angle by geometrical methods,) consists in its being the projection of the curve called the Helix; that is, of the thread of the common mechanical screw, upon a plane coinciding with the axis of the screw. But this belongs to the province of solid geometry.

The Conchoid of Nicomedes.

262. Def.—*The* Conchoid of Nicomedes *is a plane curve, such that every point of it is equi-distant*

from a given right line, the distance being measured along the right line which joins that point and a given or fixed point.

263.—*To find the equations of the conchoid.* (Fig. 70.)

Let S be the fixed point, O X the given right line which make the axis of x, S O Y being that of y; then P being any point (x, y) of the curve, we have $PQ =$ constant distance $= a$ suppose. Also let $SO = b$, which is given.

Now, $PQ^2 = PM^2 + QM^2$, or $a^2 = y^2 + QM^2$,

and $QM : y :: PN : NS :: x : b + y$;

$$\therefore \quad a^2 = y^2 + \frac{x^2 y^2}{(b + y)^2};$$

$$\text{or, } x^2 = \frac{(a^2 - y^2)\ (b + y)^2}{y^2}.$$

If the constant distance be measured in the opposite direction as QP', then $N'S = b \sim y$, and the equation becomes

$$x^2 = \frac{(a^2 - y^2)\ (b \sim y)^2}{y^2}.$$

When b is less than a, the under branch of the curve has an oval part, as in the diagram, which is called a *nodus*. (See Fig. 71.)

The polar equation referred to S as pole, ($\theta = OSP$) is $r = b \sec. \theta \pm a$.

It is easy to find the equation of the tangent at any point of this curve, as in the case of the conic sections, and thence to show that the axis of x is an asymptote to the curve, which is effected by supposing the asymp-

tote to be a tangent at a point whose co-ordinate of x is infinite. If the equation of the tangent be $\frac{x'}{A} + \frac{y'}{B}$ at the point (x, y) it will thus be very easily found that

$$A = x + \frac{2y^3 - 3by^2 - (a^2 - b^2)y + x^2 y - a^2 b}{xy}$$

$$B = y + \frac{x^2 y^2}{2y^3 - 3by^2 - (a^2 - b^2)y + x^2 y - a^2 b}$$

and putting $x = \infty$; then by the equation, $y = 0$ and

$$A = x + \frac{x^2 y}{xy} = 2x = 2\infty = \infty$$

$$B = 0 + \frac{x^2 y^2}{x^2 y} = 0 + y = 0.$$

$\therefore$ the equation of the tangent at $(\infty\ O,)$ is

$$\therefore \quad \frac{x'}{\infty} + \frac{y'}{0} 1\, y' = 0$$

is that equation which belongs to the axis of x.

Nicomedes applied this curve to the solution of the famous problems *The Trisection of an Angle and the Duplication of the Cube;* for a full account of which the reader may consult Montucla's 'Histoire des Mathematiques,' tom. i. p. 236, or Bossut's 'History of the Mathematics' (translation); or Barlow's or Hutton's 'Mathematical Dictionary'

The Cissoid of Diocles.

264. Def.—The Cissoid *is a plane curve any point of which is in the chord of a circle always passing through a given point of the circle, and such that a*

perpendicular being drawn upon the diameter passing through the given point shall be equidistant from the centre with a $\perp$ upon the diameter from the extremity of the chord. (Fig. 72.)

265.—*To find the equation of the Cissoid.*

Let P be any point (x, y) of the cissoid, O Q being the chord of the circle always passing through O, and O A the diameter, which take for the axis of x. Also let Q N be $\perp$ O A. Then by the definition

$$x = \mathrm{OM} = \mathrm{AN} = 2\mathrm{R} - \mathrm{ON},$$

and $y = x \cdot \dfrac{\mathrm{QN}}{\mathrm{ON}} = x \cdot \dfrac{\sqrt{(2\mathrm{R} \cdot \mathrm{ON} - \mathrm{ON}^2)}}{\mathrm{ON}}$;

$$\therefore\ y^2 = x^2 \cdot \left(\frac{2\mathrm{R}}{\mathrm{ON}} - 1\right) = x^2 \left(\frac{2\mathrm{R}}{2\mathrm{R} - x} - 1\right),$$

or $y^2 = \dfrac{x^3}{2\mathrm{R} - x}$;

the equation required.

The polar equation, r being $= \mathrm{OP}$ and $\theta = \angle \mathrm{COP}$, is

$$r = \frac{2\mathrm{R}\sin.^2\theta}{\cos.\theta}.$$

For $y = r\sin.\theta,\ x = r\cos.\theta$;

$$\therefore\quad r^2\sin.^2\theta = \frac{r^3\cos.^3\theta}{2\mathrm{R} - r\cos.\theta};$$

$$\therefore\quad 2\mathrm{R}\sin.^2\theta - r\cos.\theta\sin.^2\theta = r\cos.^3\theta;$$

$$\therefore\quad r = \frac{2\mathrm{R}\sin.^2\theta}{\cos.\theta\sin.^2\theta + \cos.^3\theta}$$

$$= 2\mathrm{R}\frac{\sin.^2\theta}{\cos.\theta}.$$

This curve was also used for the trisection of an angle

and for the duplication of the cube. See the authors quoted in (Art. 263).

THE LOGARITHMIC CURVE.

266. DEF.—*The* LOGARITHMIC CURVE *is that in which one co-ordinate is the logarithm of the other.*

267.—*To find the equation of the logarithmic curve.*

Let a be the base of the system of the logarithms employed; then

$x = \log. y$ in the system whose base is a, or $y = a^x$.

There is no particular use to which this curve can be applied; but it possesses some curious properties with respect to its tangent and area.

268.—THE HARMONIC CURVE.

A branch of it has the equation

$$y = a \cos.^{-1}\left(1 - \frac{m}{a}x\right).$$

It results from a very delicate investigation in the theory of the vibration of cords, in Mechanics.

SECTION XII.

CURVES WHICH ARE USEFUL ONLY IN EXEMPLIFICATION OF THE DIFFERENTIAL CALCULUS.

This class of curves we shall define merely by giving their equations, it being a waste of the student's time to consider their definitions as founded upon their geometrical properties, as also those and all other properties they severally possess. We should have omitted them

altogether, like some other authors, but for the illustration they afford to that application of the Differential and Integral Calculus which relates to the general theory of curves.

269.—*The* SEMI-CUBIC PARABOLA *is defined by the equation*

$$a\,y^2 = x^3.$$

270.—The CUBIC PARABOLA by

$$a^2\,y = x^3.$$

271.—The GENERAL PARABOLA by

$$y = a + b\,x + c\,x^2 + d\,x^3 + \ldots$$

to any given number of terms.

272.—The QUADRATRIX OF DINOSTRATUS by

$$y = (a - x)\tan.\frac{\pi}{2}.\frac{x}{a}.$$

273.—The TRACTORY by

$$x + \sqrt{(a^2 - y^2)} = a\log.\frac{a + \sqrt{(a^2 - y^2)}}{y}.$$

274.—The SYNTRACTORY OF RICCATI by

$$x + \sqrt{(a^2 - y^2)} = b\log.\frac{a + \sqrt{(a^2 - y^2)}}{y}.$$

275.—The LEMNISCATA OF JAMES BERNOUILLI by

$$(x^2 + y^2)^2 = a^2(x^2 - y^2).$$

276.—The WITCH OF DONNA AGNESI by

$$y = \frac{a\sqrt{(a\,x - x^2)}}{x}.$$

277.—The SPIRAL OF ARCHIMEDES by

$$\sqrt{(x^2 + y^2)} = a\,.\cos.^{-1}\frac{x}{\sqrt{(x^2 - y^2)}},$$

or more simply, in polar co-ordinates, by

$$r = a\,\theta.$$

Its geometrical definition is hence evident.

278. DEF.—A *Spiral* is a curve which passes in its course an indefinite number of times about a certain point within it called the *Pole of the Spiral.*

279.—The LOGARITHMIC or EQUIANGULAR SPIRAL, by

$$\cos.^{-1}\frac{x}{\sqrt{(x^2+y^2)}} = \frac{b}{\sqrt{(a^2-b^2)}}\log.\frac{\sqrt{(x^2+y^2)}}{a},$$

or by $\theta = \dfrac{b}{\sqrt{(a^2-b^2)}}\log.\dfrac{r}{a},$

or by $p = \dfrac{b}{a}\,r;$

p being the $\perp$ on the tangent.

Its geometrical definition is evident from the last equation; which definition is implied by the term *equiangular* spiral.

280.—COTES' SPIRALS, by

$$p = \frac{b\,r}{\sqrt{(a^2 \pm r^2)}};$$

which are deduced by him as those orbits of the planets which would result from the law of universal gravitation being the inverse cube instead of the inverse square of the distance of the sun from the planet.

When $a = b$, and the positive sign is taken, the polar equation is

$$r = \frac{a}{\theta};$$

$$\therefore \quad \sqrt{(x^2+y^2)}\cos.^{-1}\frac{x}{\sqrt{(x^2+y^2)}} = a;$$

and in this case the curve is called the *Hyperbolic Spiral.*

When a is $> b$, the polar equation is

$$\theta = \frac{b}{\sqrt{(a^2 - b^2)}} \log. \frac{\sqrt{(r^2 + a^2 - b^2)} - \sqrt{(a^2 - b^2)}}{r}.$$

When a is $> b$,

$$\theta = \frac{b}{\sqrt{(b^2 - a^2)}} \sec.^{-1} \frac{r}{\sqrt{(b^2 - a^2)}},$$

or $$r = \sqrt{(b^2 - a^2)} \sec. \frac{\sqrt{(b^2 - a^2)}}{b} \theta.$$

When the negative sign is taken,

$$\theta = \frac{b}{\sqrt{(a^2 - b^2)}} . \log. \frac{\sqrt{(a^2 - b^2)} + \sqrt{(a^2 - b^2 - r^2)}}{r}.$$

In all which cases the rectangular equation will be had from substituting $\cos.^{-1} \dfrac{x}{\sqrt{(x^2 + y^2)}}$ for θ, and $\sqrt{(x^2+y^2)}$ for r.

281.—The Lituus by

$$a^2 + y^2 = \frac{a^2}{\cos.^{-1} \dfrac{x}{\sqrt{(x^2 + y^2)}}},$$

or by $$r^2 = \frac{a^2}{\theta},$$

or by $$p = 4\, a^2 . \frac{r}{\sqrt{(a^4 + r^4)}}.$$

As it has before been observed, the diversity of curves is absolutely infinite. In fact every *one* equation of *two* unknown quantities is the equation of some curve or other, the unknown quantities being, for want of another

simultaneous equation involving them and given quantities, *variables*, and consequently the co-ordinates of any point in a certain curve.

Section XIII.

GENERAL PROPERTIES OF CURVES.

282. Prop.—*Either co-ordinate of a curve whose equation is of* n *dimensions, has* n, *or* n — 2, or n — 4, &c. *different values for any given value of the other co-ordinate, unless some one or more of these values coincide and touch the curve.*

For the general equation between x and y of n dimensions is of the form

$$y^n + (a x + b)\, y^{n-1} + (c x^2 + d x + e)\, y^{n-2} + \&c.$$
$$+ A x^n + B x^{n-1} + \&c. = 0;$$

and for every value of x, there are n or $n - 2$, $n - 4$, &c. values of y, according as the equation has no imaginary roots, or two, or four, or six, &c., provided that none of the real roots are equal. When the real roots are any of them equal, the co-ordinate will have so many the less of different values.

The same may be shown with respect to x for any given value of y.

283. Prop.—*Every curve, whose equation is of an odd number of dimensions, has, at least, one infinite branch on each side of the origin of co-ordinates.*

For any value whatever, and therefore ∞ being assumed for one co-ordinate, the resulting equation being

of an odd number of dimensions, will give at least, one real value of the other co-ordinate.

284. PROP.—*The sum of all the values of either co-ordinate with their signs changed corresponding to any* one *given value of their other co-ordinate, is the co-efficient of the second term of the resulting equation of the curve; the sum of the products of every two of them is the co-efficient of the third term; that of every three of them the co-efficient of the fourth term, &c.; and the product of all of them is the last term, their signs being changed in every case.*

For $ax + b$ is the sum of the values of $- y$, or roots of the equation;

$cx^2 + dx + e$ the sum of the products of every two, &c.

and $Ax^n + Bx^{n-1} + \&c.$ the product of all the values of $- y$.

285. PROP.—*If any right line bisect two parallel chords of a curve it is a diameter of the curve; that is, it bisects all chords parallel to them.* (Fig. 73.)

Let the equation of the curve be $y^n + (ax + b)y^{n-1} + \&c. = 0$, and let O N bisect the parallel chords pp', qq' making

$$PM = P'M,\ pM = p'M$$
$$QN = Q'N,\ qN = q'N.$$

Then O being the origin of co-ordinates and O X, O Y parallel to the chords, the axes, since the sum of the negative = sum of the positive ordinates, we have

$$\left.\begin{array}{r} a \,.\, OM + b = 0 \\ \text{and } a \,.\, ON + b = 0 \end{array}\right\} \begin{array}{l} \therefore\ a\,.\,(OM - ON) = 0; \\ \therefore\ a = 0, \text{ and hence } b = 0; \end{array}$$

$$\therefore \quad a x + b = o;$$

that is, the sum of the positive ordinates = the sum of the negative ones at any point whatever of the curve.

286. Cor.—Hence *the locus of the middle points of any number of parallel chords in any of the conic sections* (See Hamilton's 'Conic Sections,' Arts. 74, 126, &c.) *is a right line.*

287. Prop.—*If two right lines each cut any curve in as many points as its equation has dimensions, the continued product of the distances of those points in one of the lines, from the intersection of the lines, is to that of the distances of those points in the other line from the same intersection, in a given ratio.* (Fig. 74.)

Those who wish to pursue this interesting, though useless, branch further, may consult Euler's 'Analysis Infinitorum,' or Waring's 'Proprietates Algebraicarum Curvarum.'

Let the given right lines X X′, Y Y′ be the axes of co-ordinates, and their intersection 0 the origin of co-ordinates; and suppose the equation of the curve to be

$$y^{n} + (a x + b) y^{n-1} + \ldots . A x^{n} + B x^{n-1} + \&c. + L = 0.$$

Then whatever is the value of x the term $A x^{n} + B x^{n-1} + \&c.$ = product of all the values of $- y$.

Let $x = 0$; then

$$L = \pm OB . Ob . OB' . Ob' \&c.$$

Again, let $y = 0$; then

$$x^{n} + \frac{B}{A} x^{n-1} + \ldots . + \frac{L}{A} = 0,$$

$$\text{and } \frac{L}{A} = \pm\, OA \,.\, Oa \,.\, OA' \,.\, Oa', \text{ \&c.}$$

$$\therefore\ OA \,.\, Oa \,.\, OA' \,.\, Oa' \text{ \&c.} = OB \,.\, Ob \,.\, OB' \,.\, Ob' \text{ \&c.} \times A.$$

288. Cor. 1.—*Hence in every conic section if two chords intersect one another either within or without the curve, the product of the distances of the intersection from the extremities of one chord is to that of the distances from the extremities of the other, in a given ratio.* (Fig. 75.)

The ratio is one of equality in the case of the circle; for $A = 1$.

289. Cor. 2.—*If from the same point a tangent and chord be drawn to a conic section, the square of the tangent is to the product of the distances of the point from the extremities of the chord in a given ratio.*

290. Cor. 3.—*If from the same point two tangents be drawn to any conic section, the square of the one is to the square of the other in a given ratio.*

Formulæ for Memory.

In the Theory of a Point.

1. *The distance of the points (a, b), (a', b') is (rectangular co-ordinates)*

$$\sqrt{\{(a-a')^2 + (b-b')^2\}}.$$

2. *The same when the axes make the angle α, is*

$$\sqrt{\{(a-a')^2 + (b-b')^2 + 2\,(a-a')\,(b-b')\cos.\alpha\}}.$$

In the Theory of a Right Line.

3. *The equation of a right line is of the form*

$$\frac{x}{a} + \frac{y}{b} = 1,$$

in which a, b are the common ordinates of it and the axes of x and y respectively; and $\frac{b}{a}$ = tangent of the angle that the line makes with the axis of x.

$$\text{or } y = Ax + B,$$

in which A is the tangent of the angle between the line and the axis of x;

$$\text{or } x = A'y + B'.$$

But the first is the best form, being symmetrical and homogeneous.

4. *The polar equation of a right line is*

$$\frac{r \cos. \theta}{a} + \frac{r \sin. \theta}{b} = 1.$$

5. *The angle between two right lines* $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{a'} + \frac{y}{b'} = 1$, *is*

$$\tan.^{-1} \frac{\frac{b}{a} - \frac{b'}{a'}}{1 + \frac{b}{a} . \frac{b'}{a'}}$$

6. *The condition that two right lines be parallel*, viz. $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{a'} + \frac{y}{b'} = 1$ *is*

$$\frac{b}{a} = \frac{b'}{a'}.$$

7. *The condition that they be at right angles is*

$$\frac{b'}{a'} = -\frac{1}{\left(\frac{b}{a}\right)}.$$

8. *The length of the perpendicular drawn from a given point* (a, b) *upon the right line* $\frac{x}{a'} + \frac{y}{b'} = 1$ *is*

$$\frac{\frac{a}{a'} + \frac{b}{b'} - 1}{\sqrt{\left(\frac{1}{a'^2} + \frac{1}{b'^2}\right)}}.$$

In the Theory of a Circle.

9. *The equation of a circle whose centre is* (a, b) *and radius* R *is*

$$(x - a)^2 + (y - b^2) = R^2.$$

10. *The polar equation of a circle whose centre is* (R, α) *and radius* S *is*

$$r^2 - 2 R r \cos. (\theta - \alpha) = S^2 - R^2.$$

11. *The equation of a circle, origin at the centre and axis of* x *a diameter, is*

$$x^2 + y^2 = R^2.$$

Origin at extremity of a diameter, the axis of x being the diameter, is

$$x^2 + y^2 = 2 R x.$$

12. *The general form of the equation of a circle is*

$$x^2 + y^2 + A x + B y + C = 0.$$

13. *The general form of the polar equation of a circle is*

$$r^2 + A . r \cos. (\theta - \alpha) + B = 0.$$

14. *The equation of the tangent of a circle* $x^2 + y^2 = R^2$ *at any point* (a, b) *of it, is*

$$\frac{x}{\left(\frac{R^2}{a}\right)} + \frac{y}{\left(\frac{R^2}{b}\right)} = 1.$$

15. *The equation of the normal at the same point, is*

$$\frac{x}{a} - \frac{y}{b} = 0.$$

16. *The area of a circle is* πR^2, π being $= 3.14159$, &c. *Its circumference* $= 2\pi R$.

In the Theory of the Parabola.

17. *The equation of the parabola* (origin at vertex and axis of x the axis of curve) *is*

$$y^2 = 4Sx.$$

18. *The same when origin is any where in the curve and axis of* x *parallel to the directrix, is*

$$x^2 - 2ax + 2\{\sqrt{(a^2 + b^2)} - b\}y = 0.$$

19. *The general equation of a parabola is*

$$\left(\frac{x}{A} + \frac{y}{B}\right)^2 + \frac{x}{C} + \frac{y}{D} + 1 = 0.$$

20. *The polar equation of the parabola, pole being the focus and* θ *measured from the vertex*

$$r = \frac{2S}{1 + \cos.\theta} \text{ or } \frac{S}{\cos.^2\frac{\theta}{2}}.$$

21. *That when* θ *is measured from a right line which makes the angle* α *with the axis of the parabola, is*

$$r = \frac{2S}{1 + \cos.(\alpha + \theta)}.$$

22. *The equation of the tangent at any point* (a, b) *of the parabola* $y^2 = 4Sx$,

$$\frac{x}{(-a)} + \frac{y}{\left(\frac{b}{2}\right)} = 1.$$

23. *The equation between* r *and* p *referred to the focus of the parabola, is*

$$p^2 = \text{S}\, r.$$

24. *The equation of the normal of the parabola at any point* (a, b), *is*

$$\frac{x}{a + 2\text{S}} + \frac{y}{\frac{2a}{b}(a + 2\text{S})} = 1.$$

25. The radius of curvature of the parabola, is

$$r = 2(\text{S} + a)\sqrt{\frac{\text{S} + a}{\text{S}}}\,.$$

26. *The area of any part of a parabola cut off by ordinates, is* $\frac{2}{3}$ *of its circumscribing rectangle.*

27. *Lambert's theorem is*

Sectorial area of parabola

$$= \frac{\sqrt{\text{S}}}{6\sqrt{2}}\left\{(\text{R}' + \text{R} + c) - (\text{R}' + \text{R} - c)^{\frac{3}{2}}\right\}.$$

In the Theory of the Ellipse.

28. *Equation of the ellipse, origin in centre and axis of* x *the focal axis*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

a *and* b *being the semi-focal and semi-non-focal axes.*

29. *Equation when origin is at vertex*, &c.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2\cdot\frac{x}{a}\,.$$

30. *General Equation of an ellipse is that of two*

dimensions, in which the co-efficients of x^2 *and* y^2 *have the same sign.*

31. *The polar equation of the ellipse referred to focus, is*

$$r = \frac{a\,(1 - e^2)}{1 + e\cos.\,(\theta - \alpha)}.$$

32. *The polar equation of the ellipse referred to its centre, is*

$$r^2 = \frac{a(1 - e^2)}{1 - e^2\cos.^2\,(\theta - \alpha)}.$$

33. The equation of the tangent at any point (a, b) of the ellipse $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$, is

$$\frac{x}{\left(\frac{a'^2}{a}\right)} + \frac{y}{\left(\frac{b'^2}{b}\right)} = 1.$$

34. *The equation of the normal at the same point is*

$$\frac{x}{a\left(1 - \frac{b'^2}{a'^2}\right)} + \frac{y}{b\left(1 - \frac{a'^2}{b'^2}\right)} = 1.$$

35. *The equation between* p *and* r *(referred to focus) is*

$$p^2 = \frac{b'^2\,r}{2\,a' - r}.$$

36. *That referred to centre is*

$$p^2 = \frac{a'^2\,b'^2}{a'^2 + b'^2 - r^2}.$$

37. *The radius of curvature at any point* (a, b) *of the ellipse* $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$, *is*

$$\rho = \frac{(a'^2 + b'^2 - a^2 - b^2)^{\frac{3}{2}}}{a'\,b'}.$$

38. The chord of curvature through the centre of the ellipse, is

$$PV = 2 \cdot \frac{a'^2 + b'^2 - a^2 - b^2}{\sqrt{(a^2 + b^2)}}.$$

39. That through the focus is

$$PV' = 2 \cdot \frac{a'^2 + b'^2 - a^2 - b^2}{a'}.$$

40. *The whole area of an ellipse is*

$$\pi \cdot a b.$$

IN THE THEORY OF THE HYPERBOLA.

41. *Equation of the hyperbola,* (origin the centre) *is*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

42. *The same (origin at the vertex) is*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -2 \cdot \frac{x}{a}.$$

43. *The general equation of the hyperbola is that of two dimensions in which the co-efficients of* x^2 *and* y^2 *have different signs.*

44. *The polar equations of the hyperbola,* (pole the focus) *is*

$$r = \frac{a(e^2 - 1)}{1 + e \cos. (\theta - \alpha)}.$$

45. *The same when the pole is the centre is*

$$r^2 = \frac{a^2 (e^2 - 1)}{e^2 \cos.^2 (\theta - \alpha) - 1}.$$

46. *The equations of the asymptotes of the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *are*

$$\frac{y}{b} = \pm \frac{x}{a}.$$

47. *The equation of the hyperbola referred to its asymptotes is*

$$x\,y = \frac{a^2 + b^2}{4}.$$

48. *The equation of the tangent at any point* (a, b) *of the hyperbola* $\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1$ *is*

$$\frac{x}{\left(\frac{a'^2}{a}\right)} - \frac{y}{\left(\frac{b'^2}{b}\right)} = 1.$$

49. *That of the normal is*

$$\frac{x}{a\left(1 + \frac{b'^2}{a'^2}\right)} + \frac{y}{b\left(1 + \frac{a'^2}{b'^2}\right)} = 1.$$

50. *The equation between* p *and* r (pole the focus) *is*

$$p^2 = \frac{b'^2\,r}{2\,a' + r}.$$

51. *The same when the pole is the centre is*

$$p^2 = \frac{a'^2\,b'^2}{r^2 - a'^2 + b'^2}.$$

52. *The radius of curvature is*

$$\rho = \frac{(a^2 + b^2 - a'^2 + b'^2)^{\frac{3}{2}}}{a'\,b'}.$$

53. *The chord through the centre is*

$$2\,.\,\frac{a^2 + b^2 - a'^2 + b'^2}{\sqrt{(a^2 + b^2)}}.$$

54. *The chord through the focus is*

$$2 \cdot \frac{a^2 + b^2 - a'^2 + b'^2}{a'}.$$

55. *The general equation of conic sections is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 \cdot \frac{x}{a}.$$

IN THE TRANSPOSITION OF CO-ORDINATES.

56. *The rectangular co-ordinate axe of any point* (x, y) *may be changed to other rectangular co-ordinate axes whose equations are*

$$\frac{x}{a} + \frac{y}{b} = 1, \frac{x}{a'} + \frac{y}{b'} = 1$$

by substituting for x *and* y *their values in*

$$\frac{x}{a} + \frac{y}{b} = 1 + \frac{y'}{b} \frac{\sqrt{(a^2 + b^2)}}{a}$$

$$\frac{x}{a'} + \frac{y}{b'} = 1 - \frac{x'}{a} \frac{\sqrt{(a'^2 + b'^2)}}{b'}.$$

57. The same may be done when the axes are oblique, by substituting the values of (x, y) in

$$\frac{x}{a} + \frac{y}{b} = 1 + \frac{y'}{b} \frac{\sin.\,(x'\,y')}{\sin.\,(x'\,y)}$$

$$\frac{x}{a'} + \frac{y}{b'} = 1 + \frac{x'}{b'} \cdot \frac{\sin.\,(x'\,y')}{\sin.\,(y'\,y)}.$$

58. *Equation of curves may be simplified by assuming*

$x = \mathrm{A} + x' \cos.\,\theta + y' \sin.\,\theta$

$y = \mathrm{B} + x' \sin.\,\theta + y' \cos.\,\theta$, &c. &c.

59. *The general equation*

$$\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}xy + \mathrm{D}x + \mathrm{E}y + \mathrm{F} = 0$$

is that of a right line, a circle, a parabola, an ellipse or hyperbola, according as

$$\left.\begin{array}{r} C^2-4AB=0 \\ CD-2AE=0 \end{array}\right\}; \text{ and } \left.\begin{array}{r} C=0 \\ A=B \end{array}\right\}; \; C^2-4AB=0;$$

$C^2 - 4AB$ is negative; or $C^2 - 4AB$ positive.

In the Higher Curves.

60. *The equation of a cycloid is*

$$x = \sqrt{(2Ry - y^2)} + \cos.^{-1}\frac{R-y}{R}.$$

61. *That of the trochoid of Newton is*

$$x = \cos.^{-1}\frac{R-y}{R'} - \sqrt{\{R'^2 - (y-R)^2\}}.$$

62. *The equation of the companion of the cycloid is*

$$y = R . \cos.^{-1}\frac{R-x}{R}.$$

63. *That of the catenary is*

$$y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}}) - a.$$

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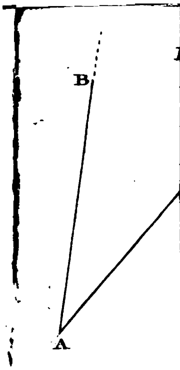

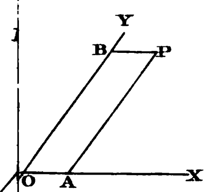

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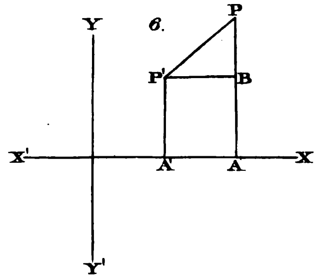

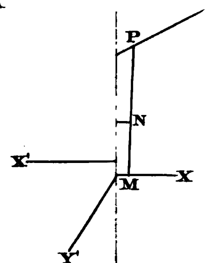

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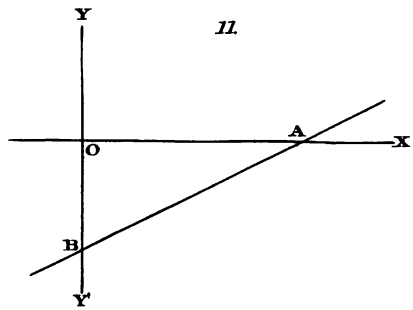

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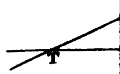

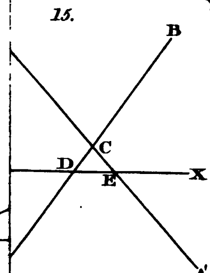

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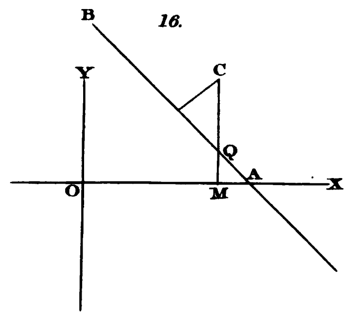

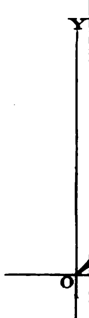

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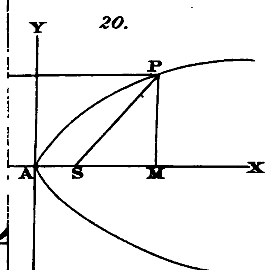

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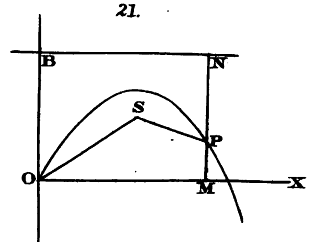

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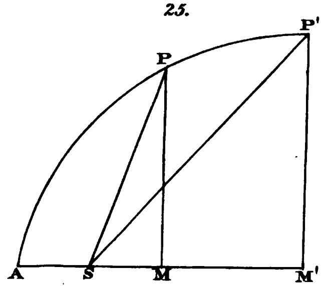

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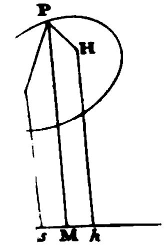

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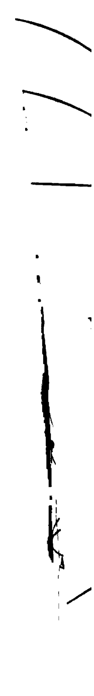

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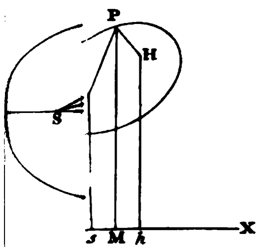

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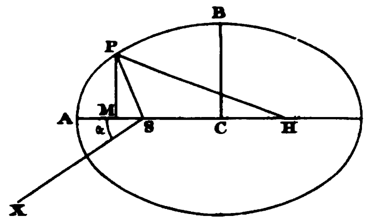

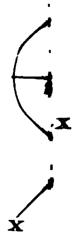

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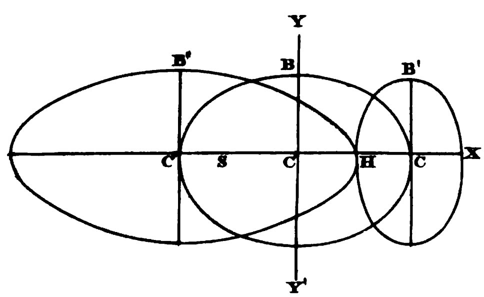

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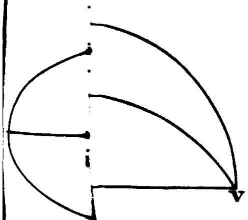

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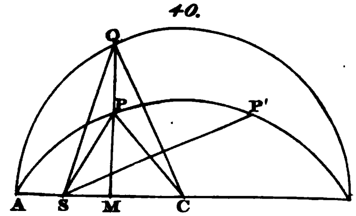

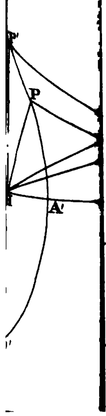

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